UTILITY FUNCTIONS: FROM RISK THEORY TO FINANCE

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ABSTRACT

This article is a self-contained survey of utility functions and some of their applications. Throughout the paper the theory is illustrated by three examples: exponential utility functions, power utility functions of the first kind (such as quadratic utility functions), and power utility functions of the second kind (such as the logarithmic utility function). The postulate of equivalent expected utility can be used to replace a random gain by a fixed amount and to determine a fair premium for claims to be insured, even if the insurer's wealth without the new contract is a random variable itself. Then *n* companies (or economic agents) with random wealth are considered. They are interested in exchanging wealth to improve their expected utility. The family of Pareto optimal risk exchanges is characterized by the theorem of Borch. Two specific solutions are proposed. The first, believed to be new, is based on the synergy potential; this is the largest amount that can be withdrawn from the system without hurting any company in terms of expected utility. The second is the economic equilibrium originally proposed by Borch. As by-products, the option-pricing formula of Black-Scholes can be derived and the Esscher method of option pricing can be explained.

1. INTRODUCTION

The notion of utility goes back to Daniel Bernoulli (1738). Because the value of money does not solve the paradox of St. Petersburg, he proposed the *moral value* of money as a standard of judgment. According to Borch (1974, p. 26),

> Several mathematicians, for example Laplace, discussed the Bernoulli principle in the following century, and its relevance to insurance systems seems to have been generally recognized. In 1832 Barrois presented a fairly complete theory of fire insurance, based on Laplace's work on the Bernoulli principle. For reasons that are difficult to explain, the principle was almost completely forgotten, by actuaries and economists alike, during the next hundred years.

This is confirmed by Seal (1969, Ch. 6) and the references cited therein.

Utility theory came to life again in the middle of this century. This was above all the merit of von Neumann and Morgenstern (1947), who argued that the existence of a utility function could be derived from a set of axioms governing a preference ordering. Borch showed how utility theory could be used to formulate and solve some problems that are relevant to insurance. Due to him, risk theory has grown beyond ruin theory. Most of the original papers of Borch have been reprinted and published in book form (1974, 1990).

Economic ideas have greatly stimulated the development of utility theory. But this also means that substantial parts of the literature have been written in a style that does not appeal to actuaries.

The purpose of this paper is to give a concise but self-contained survey of utility functions and their applications that might be of interest to actuaries. In Sections 2 and 3 the notion of a utility function with its associated risk aversion function is introduced. Throughout the paper, the theory is illustrated by means of examples, in which exponential utility functions and power utility functions of the first and

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second kind are considered, which include quadratic and logarithmic utility functions.

In Section 4, we order random gains by means of their expected utilities. In particular, a random gain can be replaced by a fixed amount, the certainty equivalent. This notion can be used by the consumer who wants to determine the maximal premium he or she is willing to pay to obtain full coverage.

The insurer's situation is considered in Section 5. A premium that is fair in terms of expected utility typically contains a loading that depends on the insurer's risk aversion and on the joint distribution of the claims, S, and the random wealth, W, without the new contract; certain rules of thumb in terms of $\mathrm{Var}[S]$ and $\mathrm{Cov}(S, W)$ are obtained. Section 6 presents a classical result that can be found as Theorem 1.5.1 in Bowers et al. (1997).

In Section 8, we consider n companies with random wealth. Can they gain simultaneously by trading risks? The class of Pareto optimal exchanges is discussed and characterized by the theorem of Borch. Two more specific solutions are proposed. The first idea is to withdraw the synergy potential, which is the largest amount that can be withdrawn from the system of the n companies without hurting any of them. Then this amount is reallocated to the companies in an unambiguous fashion. The second idea, as presented by Bühlmann (1980, 1984), is to consider a competitive equilibrium, in which random payments can be bought in a market. Here the equilibrium price density plays a crucial role. In Section 11 it is shown how options can be priced by means of the equilibrium price density. This approach differs from chapter 4 of Panjer et al. (1998), which considers the utility of consumption and assumes the existence of a representative agent.

2. UTILITY FUNCTIONS

Often it is not appropriate to measure the usefulness of money on the monetary scale. To explain certain phenomena, the usefulness of money must be measured on a new scale. Thus, the usefulness of \$$x$ is $u(x)$, the *utility* (or "moral value") of \$$x$. Typically, x is the wealth or a gain of a decision-maker.

We suppose that a utility function $u(x)$ has the following two basic properties:

(1) $u(x)$ is an increasing function of x

(2) $u(x)$ is a concave function of x.

Usually we assume that the function $u(x)$ is twice differentiable; then (1) and (2) state that $u'(x) > 0$ and $u''(x) < 0$.

The first property amounts to the evident requirement that more is better. Several reasons are given for the second property. One way to justify it is to require that the marginal utility $u'(x)$ be a decreasing function of wealth x, or equivalently, that the gain of utility resulting from a monetary gain of \$$g$, $u(x + g) - u(x)$, be a decreasing function of wealth x.

Example 1

Exponential utility function (parameter $a > 0$)

$$u(x) = \frac{1}{a}(1 - e^{-ax}), \qquad -\infty < x < \infty. \tag{1}$$

We note that for $x \to \infty$, the utility is bounded and tends to the finite value $1/a$.

Example 2

Power utility function of the first kind (parameters $s > 0$, $c > 0$)

$$u(x) = \frac{s^{c+1} - (s - x)^{c+1}}{(c + 1)s^c}, \qquad x < s. \tag{2}$$

Obviously this expression cannot serve as a model beyond $x = s$. The only way to extend the definition beyond this point so that $u(x)$ is a nondecreasing and concave function is to set $u(x) = s/(c + 1)$ for $x \geq s$. In this sense s can be interpreted as a level of saturation: the maximal utility is already attained for the finite wealth s. The special case $c = 1$ is of particular interest. Then

$$u(x) = x - \frac{x^2}{2s}, \qquad x < s \tag{3}$$

is a quadratic utility function.

Example 3

Power utility function of the second kind (parameter $c > 0$). For $c \neq 1$ we set

$$u(x) = \frac{x^{1-c} - 1}{1 - c}, \qquad x > 0. \tag{4}$$

For $c = 1$ we define

$$u(x) = \ln x, \qquad x > 0. \tag{5}$$

Note that (5) is the limit of (4) as $c \to 1$.

Remark 1

A utility function $u(x)$ can be replaced by an equivalent utility function of the form

$$\tilde{u}(x) = Au(x) + B \tag{6}$$

(with $A > 0$ and B arbitrary). Hence it is possible to standardize a utility function, for example, by requiring that

$$u(\xi) = 0, \qquad u'(\xi) = 1 \tag{7}$$

for a particular point ξ. In Examples 1 and 2 this has been done for $\xi = 0$; in Example 3 it has been done for $\xi = 1$.

Remark 2

If we take the limit $a \to 0$ in Example 1, or $s \to \infty$ in Example 2, we obtain $u(x) = x$, a linear utility function, which is not a utility function in the proper sense. Similarly, the limit $c \to 0$ in Example 3 is $u(x) = x - 1$.

Remark 3

In the following we tacitly assume that $x < s$ if $u(x)$ is a power utility function of the first kind, and that $x > 0$ if $u(x)$ is a power utility function of the second kind. The analogous assumptions are made when we consider the utility of a random variable.

3. RISK AVERSION FUNCTIONS

To a given utility function $u(x)$ we associate a function

$$r(x) = \frac{-u''(x)}{u'(x)} = -\frac{d}{dx}\ln u'(x), \tag{8}$$

called the *risk aversion function*. We note that properties (1) and (2) imply that $r(x) > 0$. Let us revisit the three examples of Section 2.

For the *exponential utility function* (parameter $a > 0$), we find that

$$r(x) = a, \qquad -\infty < x < \infty. \tag{9}$$

Thus the exponential utility function yields a constant risk aversion.

For the *power utility function of the first kind* (parameters $s > 0$, $c > 0$), we find that

$$r(x) = \frac{c}{s - x}, \qquad x < s. \tag{10}$$

Here the risk aversion increases with wealth and becomes infinite for $x \to s$; this has the following interpretation: if the wealth is close to the level of saturation s, very little utility can be gained by a monetary gain; hence there is no point in taking any risk.

For the *power utility function of the second kind* (parameter $c > 0$), we obtain

$$r(x) = \frac{c}{x}, \qquad x > 0. \tag{11}$$

Here the risk aversion is a decreasing function of wealth, which may be typical for some investors.

If $u(x)$ is replaced by an equivalent utility function as in (6), the associated risk aversion function is the same. In the opposite direction, if we are given the risk aversion function $r(x)$ and want to find an underlying utility function, we look for a function $u(x)$ that satisfies the equation

$$u''(x) + r(x)u'(x) = 0. \tag{12}$$

Such a differential equation has a two-parameter family of solutions. To get a unique answer, we may standardize according to (7) for some ξ. Then the solution is

$$u(x) = \int_{\xi}^{x} \exp\left[-\int_{\xi}^{z} r(y)dy\right] dz. \tag{13}$$

Now suppose that $r_1(x)$ and $r_2(x)$ are two risk aversion functions with

$$r_1(x) \le r_2(x) \qquad \text{for all } x. \tag{14}$$

Let $u_1(x)$ and $u_2(x)$ be two underlying utility functions. Because of their ambiguity, they cannot be compared without making any further assumptions. If we assume however, that $u_1(x)$ and $u_2(x)$ are standardized at the same point ξ, that is,

$$u_i(\xi) = 0, \qquad u_i'(\xi) = 1, \qquad i = 1, 2, \tag{15}$$

then it follows that

$$u_1(x) \ge u_2(x) \qquad \text{for all } x. \tag{16}$$

For the proof we observe that

$$u_i(x) = \int_{\xi}^{x} \exp\left[-\int_{\xi}^{z} r_i(y)dy\right] dz, \qquad \text{if } x > \xi,$$

$$u_i(x) = -\int_{x}^{\xi} \exp\left[\int_{z}^{\xi} r_i(y)dy\right] dz, \qquad \text{if } x < \xi,$$

and use the assumption (14).

4. PREFERENCE ORDERING OF RANDOM GAINS

Consider a decision-maker with initial wealth w who has the choice between a certain number of random gains. By using a utility function, two random gains can be directly compared: he or she prefers G_1 to G_2, if

$$\mathrm{E}[u(w + G_1)] > \mathrm{E}[u(w + G_2)], \qquad (17)$$

that is, if the expected utility from G_1 exceeds the expected utility from G_2. If the expected utilities are equal, he will be indifferent between G_1 and G_2. Thus a complete preference ordering is defined on the set of random gains.

If we multiply (17) by a positive constant A and add a constant B on both sides, an equivalent inequality in terms of the function $\tilde{u}(x)$ is obtained. Hence $u(x)$ and $\tilde{u}(x)$ define the same ordering and are considered to be equivalent.

Example 4

Suppose that the decision-maker uses the exponential utility function with parameter a and has the choice between two normal random variables, G_1 and G_2, with $\mathrm{E}[G_i] = \mu_i$, $\mathrm{Var}[G_i] = \sigma_i^2$, $i = 1, 2$. Since

$$\mathrm{E}[e^{-aG_i}] = \exp\left(-a\mu_i + \frac{1}{2}a^2\sigma_i^2\right),$$

it follows that

$$\mathrm{E}[u(w + G_i)] = \frac{1}{a}\left[1 - \exp\left(-aw - a\mu_i + \frac{1}{2}a^2\sigma_i^2\right)\right].$$

Hence G_1 is preferred to G_2, if (17) is satisfied, that is, if

$$\mu_1 - \frac{1}{2}a\sigma_1^2 > \mu_2 - \frac{1}{2}a\sigma_2^2. \qquad (19)$$

Jensen's inequality tells us that for any random variable G,

$$u(w + \mathrm{E}[G]) > \mathrm{E}[u(w + G)]. \qquad (20)$$

Hence, if a decision-maker can choose between a random gain G and a fixed amount equal to its expectation, he will prefer the latter. This brings us to the following definition: The *certainty equivalent,* π, associated to G is defined by the condition that the decision-maker is indifferent between receiving G or the fixed amount π. Mathematically, this is the condition that

$$u(w + \pi) = \mathrm{E}[u(w + G)]. \qquad (21)$$

From (20) we see that $\pi < \mathrm{E}[G]$. Let us consider two examples in which explicit expressions for π can be obtained:

For an *exponential utility function,* the certainty equivalent is

$$\pi = \frac{-1}{a}\ln \mathrm{E}[e^{-aG}]. \qquad (22)$$

Note that it does not depend on w. By expanding this expression in powers of a, we obtain the simple approximation

$$\pi \approx \mathrm{E}[G] - \frac{a}{2}\mathrm{Var}[G], \qquad (23)$$

valid for sufficiently small values of a.

For a *quadratic utility function,* condition (21) leads to a quadratic equation for π. Its solution can be written as follows:

$$\pi = \mathrm{E}[G] - (s - w - \mathrm{E}[G])\lambda$$

with

$$\lambda = \sqrt{1 + \frac{\mathrm{Var}[G]}{(s - w - \mathrm{E}[G])^2}} - 1. \qquad (24)$$

For large values of s, we can expand the square root and find the approximation

$$\pi \approx \mathrm{E}[G] - \frac{1}{2}\frac{\mathrm{Var}[G]}{(s - w - \mathrm{E}[G])}. \qquad (25)$$

In view of (10) we can write this formula as

$$\pi \approx \mathrm{E}[G] - \frac{1}{2}r(w + \mathrm{E}[G])\,\mathrm{Var}[G], \qquad (26)$$

which is similar to formula (23).

For a general utility function, it follows from (21) that

$$\pi = u^{-1}(\mathrm{E}[u(w + G)]) - w. \qquad (27)$$

If G is a gain with a "small" risk, the following more explicit approximation is available:

$$\pi \approx \mathrm{E}[G] - \frac{1}{2}r(w + \mathrm{E}[G])\,\mathrm{Var}[G]. \qquad (28)$$

To give a precise meaning to this statement, we set

$$G_z = \mu + zV, \qquad z > 0 \qquad (29)$$

where μ is a constant and V a random variable with $\mathrm{E}[V] = 0$ and $\mathrm{Var}[V] = \mathrm{E}[V^2] = \sigma^2$. Hence $\mathrm{E}[G_z] = \mu$ and $\mathrm{Var}[G_z] = z^2\sigma^2$. Let $\pi(z)$ be the certainty equivalent of G_z, defined by the equation

$$u(w + \pi(z)) = \mathrm{E}[u(w + G_z)]. \qquad (30)$$

The idea is to expand the function $\pi(z)$ in powers of z:

$$\pi(z) = a + bz + cz^2 + \ldots \qquad (31)$$

If we set $z = 0$ in (30), we obtain

$$u(w + a) = u(w + \mu), \qquad \text{or } a = \mu. \qquad (32)$$

If we differentiate (30), we get the equation

$$\pi'(z)u'(w + \pi(z)) = \mathrm{E}[Vu'(w + G_z)]. \qquad (33)$$

Setting $z = 0$ yields

$$bu'(w + \mu) = \mathrm{E}[V]\ u'(w + \mu) = 0,$$
$$\text{or } b = 0. \qquad (34)$$

Finally we differentiate (33) to obtain

$$\pi''(z)u'(w + \pi(z)) + \pi'(z)^2u''(w + \pi(z)) = \mathrm{E}[V^2u''(w + G_z)]. \qquad (35)$$

Setting $z = 0$ we obtain

$$2c\ u'(w + \mu) = \mathrm{E}[V^2]\ u''(w + \mu), \qquad (36)$$

or

$$c = \frac{1}{2}\frac{u''(w + \mu)}{u'(w + \mu)}\sigma^2 = -\frac{1}{2}r(w + \mu)\sigma^2. \qquad (37)$$

Substitution in (31) yields the approximation

$$\pi(z) \approx \mu - \frac{1}{2}r(w + \mu)\ z^2\sigma^2$$
$$= \mathrm{E}[G_z] - \frac{1}{2}r(w + \mathrm{E}[G_z])\mathrm{Var}[G_z], \qquad (38)$$

which explains (28).

Let us now consider two utility functions $u_1(x)$ and $u_2(x)$ so that

$$r_1(x) \le r_2(x) \qquad \text{for all } x, \qquad (39)$$

and let π_1 and π_2 denote their respective certainty equivalents. Then we expect that

$$\pi_1 \ge \pi_2. \qquad (40)$$

To verify this result, we assume that the underlying utility functions are standardized at the same point $\xi = w + \pi_1$. Then

$$u_1(x) \ge u_2(x) \qquad \text{for all } x. \qquad (41)$$

From this and the definitions of π_1 and π_2, it follows that

$$\begin{aligned} u_2(w + \pi_1) &= 0 \\ &= u_1(w + \pi_1) \\ &= \mathrm{E}[u_1(w + G)] \\ &\ge \mathrm{E}[u_2(w + G)] \\ &= u_2(w + \pi_2). \end{aligned} \qquad (42)$$

Since u_2 is an increasing function, it follows indeed that $\pi_1 \ge \pi_2$.

5. PREMIUM CALCULATION

We consider a company with initial wealth w. The company is to insure a risk and has to pay the total claims S (a random variable) at the end of the period. What should be the appropriate premium, P, for this contract? An answer is obtained by assuming a utility function, $u(x)$, and by postulating fairness in terms of utility. This means that the expected utility of wealth *with* the contract should be equal to the utility *without* the contract:

$$\mathrm{E}[u(w + P - S)] = u(w). \qquad (43)$$

This is called the *principle of equivalent utility*. Equation (43) determines P uniquely, but has no explicit solution in general. Notable exceptions are the cases in which $u(x)$ is exponential, where

$$P = \frac{1}{a}\ln \mathrm{E}[e^{aS}], \qquad (44)$$

or quadratic, where we find that

$$P = \mathrm{E}[S] + (s - w)\left\{1 - \sqrt{1 - \frac{\mathrm{Var}[S]}{(s - w)^2}}\right\}. \qquad (45)$$

If S is a "small" risk, (43) can be solved approximately as follows:

$$P \approx \mathrm{E}[S] + \frac{1}{2}r(w)\mathrm{Var}[S] \qquad (46)$$

(to see this, set $S = \mu + zV$, with $\mathrm{E}[V] = 0$, and expand P in powers of z).

In many cases a more realistic assumption is that the wealth without the new contract is a random variable itself, say W. Then P is obtained from the equation

$$\mathrm{E}[u(W + P - S)] = \mathrm{E}[u(W)]. \qquad (47)$$

Note that now P depends on the joint distribution of S and W.

Let us revisit the examples in which P can be calculated explicitly.

Example 5

If $u(x) = (1 - e^{-ax})/a$, we find that

$$P = \frac{1}{a} \ln \frac{E[e^{a(S-W)}]}{E[e^{-aW}]}. \quad (48)$$

If a is small, we can expand this expression in powers of a and obtain the approximation

$$P \approx E[S] + \frac{a}{2} \text{Var}[S - W] - \frac{a}{2} \text{Var}[W] \quad (49)$$

$$= E[S] + \frac{a}{2} \text{Var}[S] - a \,\text{Cov}(S,W). \quad (50)$$

We note that (48) reduces to (44) in the case in which S and W are independent random variables. Also, we remark that (49) is exact in the case where S and W are bivariate normal.

Example 6

If $u(x) = x - x^2/2s$, we find that

$$P = E[S] + (s - E[W])\lambda$$

with

$$\lambda = 1 - \sqrt{1 - \frac{\text{Var}[S] - 2\,\text{Cov}(S,W)}{(s - E[W])^2}}. \quad (51)$$

Note that this expression reduces to (45) with w replaced by $E[W]$, in the case in which S and W are uncorrelated random variables. For large values of s, (51) leads to the approximation

$$P \approx E[S] + \frac{1}{2} \frac{\text{Var}[S] - 2\,\text{Cov}(S,W)}{s - E[W]}$$

$$= E[S] + \frac{1}{2} r(E[W])\{\text{Var}[S] - 2\,\text{Cov}(S,W)\}. \quad (52)$$

6. Optimality of a Stop-Loss Contract

We consider a company that has to pay the total amount S (a random variable) to its policyholders at the end of the year. We compare two reinsurance agreements:

(1) A stop-loss contract with deductible d. Here the reinsurer will pay

$$(S - d)_+ = \begin{cases} S - d & \text{if } S > d \\ 0 & \text{if } S \le d \end{cases} \quad (53)$$

at the end of the year.

(2) A general reinsurance contract, given by a function $h(x)$, where the reinsurer pays $h(S)$ at the end of the year. The only restriction on the function $h(x)$ is that

$$0 \le h(x) \le x. \quad (54)$$

We assume that the two contracts are comparable, in the sense that the expected payments of the reinsurer are the same, that is, that

$$E[(S - d)_+] = E[h(S)]. \quad (55)$$

Furthermore, we make the convenient (but perhaps not realistic) assumption that the two reinsurance premiums are the same. Then, in terms of utility, the stop-loss contract is preferable:

$$E[u(w - S + h(S))] \le E[u(w - S + (S - d)_+)]. \quad (56)$$

In this context, w represents the wealth after receipt of the premiums and payment of the reinsurance premiums.

The proof of (56) starts with the observation that a concave curve is below its tangents, that is, that

$$u(y) \le u(x) + u'(x)(y - x) \qquad \text{for all } x \text{ and } y. \quad (57)$$

Using this for $y = w - S + h(S)$, $x = w - S + (S - d)_+$, we get

$$\begin{aligned} u(w-S+h(S)) &\le u(w-S+(S-d)_+) \\ &\quad + u'(w-S+(S-d)_+)(h(S)-(S-d)_+) \\ &\le u(w-S+(S-d)_+) + u'(w-d)(h(S)-(S-d)_+). \end{aligned} \quad (58)$$

To verify the second inequality, distinguish the cases $S > d$, in which equality holds, and $S \le d$, where

$$\begin{aligned} &u'(w - S + (S - d)_+)(h(S) - (S - d)_+) \\ &= u'(w - S)h(S) \\ &\le u'(w - d)h(S) \\ &= u'(w - d)(h(S) - (S - d)_+). \end{aligned}$$

Now we take expectations in (58) and use (55) to obtain (56).

7. Optimal Degree of Reinsurance

Again we consider a company that has to pay the total amount S (a random variable) at the end of the year.

A proportional reinsurance coverage can be purchased. If P is the reinsurance premium for full coverage (of course $P > E[S]$), we assume that for a premium of φP the fraction φS is covered and will be reimbursed at the end of the year ($0 \leq \varphi \leq 1$.) Then $\tilde{\varphi}$, the optimal value of φ, is the value that maximizes

$$E[u(w - \varphi P - (1 - \varphi)S)], \qquad (59)$$

where $u(x)$ is an appropriate utility function and where the initial surplus, w, includes the premiums received. In the particular case in which $u(x)$ is the exponential utility function with parameter a, and S has a normal distribution with mean μ and variance σ^2, the calculations can be done explicitly. The expected utility is now

$$\frac{1}{a}(1 - E\{\exp[-aw + a\varphi P + a(1 - \varphi)S]\})$$

$$= \frac{1}{a}\exp\left[-aw + a\varphi P + a(1-\varphi)\mu + \frac{1}{2}a^2(1-\varphi)^2\sigma^2\right].$$

It is maximal for

$$1 - \tilde{\varphi} = \frac{P - \mu}{a\sigma^2}. \qquad (60)$$

This result has an appealing interpretation. The optimal fraction that is retained is proportional to the loading contained in the reinsurance premium for full coverage, and inversely proportional to the company's risk aversion and the variance of the total claims.

In finance, a formula similar to (60) is known as the Merton ratio, see Panjer et al. (1998, Ch. 4). The difference is that for Merton's formula, the utility function is a power utility function and S is lognormal, while here the utility function is exponential and S is normal.

8. PARETO OPTIMAL RISK EXCHANGES

We consider n companies (or economic agents). We assume that company i has a wealth W_i at the end of the year and bases its decisions on a utility function $u_i(x)$. Here $W_1, \ldots, W_n$ are random variables with a known joint distribution. Let $W = W_1 + \ldots + W_n$ denote the total wealth of the companies. A risk exchange provides a redistribution of total wealth. Thus after a risk exchange, the wealth of company i will be X_i; here $X_1, \ldots, X_n$ can be any random variables provided that

$$X_1 + \ldots + X_n = W, \qquad (61)$$

that is, the total wealth remains the same. The value for company i of such an exchange is measured by

$$E[u_i(X_i)].$$

A risk exchange $(\tilde{X}_1, \ldots, \tilde{X}_n)$ is said to be *Pareto optimal,* if it is not possible to improve the situation of one company without worsening the situation of at least one other company. In other words, there is no other exchange $(X_1, \ldots, X_n)$ with

$$E[u_i(X_i)] \geq E[u_i(\tilde{X}_i)], \qquad \text{for } i = 1, \ldots, n$$

whereby at least one of these inequalities is strict. If the companies are willing to cooperate, they should choose a risk exchange that is Pareto optimal.

The Pareto optimal risk exchanges constitute a family with $n - 1$ parameters. They can be obtained by the following method: Choose $k_1 > 0, \ldots, k_n > 0$ and then maximize the expression

$$\sum_{i=1}^{n} k_i E[u_i(X_i)], \qquad (62)$$

where the maximum is taken over all risk exchanges $(X_1, \ldots, X_n)$. This problem has a relatively explicit solution:

Theorem 1 (Borch)

A risk exchange $(\tilde{X}_1, \ldots, \tilde{X}_n)$ maximizes (62) if and only if the random variables $k_i u_i'(\tilde{X}_i)$ are the same for $i = 1, \ldots, n$.

Proof

(a) Suppose that $(\tilde{X}_1, \ldots, \tilde{X}_n)$ maximizes (62). Let $j \neq h$ and let V be an arbitrary random variable. We define

$$X_i = \tilde{X}_i, \qquad \text{for } i \neq j, h,$$

$$X_j = \tilde{X}_j + tV,$$

$$X_h = \tilde{X}_h - tV,$$

where t is a parameter. Let

$$f(t) = \sum_{i=1}^{n} k_i E[u_i(X_i)]. \qquad (63)$$

According to our assumption, the function $f(t)$ has a maximum at $t = 0$. Hence $f'(t) = 0$, or

$$k_j E[V u_j'(\tilde{X}_j)] - k_h E[V u_h'(\tilde{X}_h)] = 0. \qquad (64)$$

It is useful to rewrite this equation as

$$E[V\{k_j u_j'(\tilde{X}_j) - k_h u_h'(\tilde{X}_h)\}] = 0. \qquad (65)$$

Since this holds for an arbitrary V, we conclude that

$$k_j u_j'(\tilde{X}_j) - k_h u_h'(\tilde{X}_h) = 0. \quad (66)$$

This shows indeed that $k_i u_i'(\tilde{X}_i)$ is independent of i.

(b) Conversely, let $(\tilde{X}_1, \ldots, \tilde{X}_n)$ be a risk exchange so that

$$k_i u_i'(\tilde{X}_i) = \Lambda \quad (67)$$

is the same random variable for all i. Let $(X_1, \ldots, X_n)$ be any other risk exchange. From (57) it follows that

$$u_i(X_i) \leq u_i(\tilde{X}_i) + u_i'(\tilde{X}_i)(X_i - \tilde{X}_i). \quad (68)$$

If we multiply this inequality by k_i, sum over i and use (67), we get

$$\sum_{i=1}^{n} k_i u_i(X_i) \leq \sum_{i=1}^{n} k_i u_i(\tilde{X}_i) + \Lambda \sum_{i=1}^{n} (X_i - \tilde{X}_i)$$
$$= \sum_{i=1}^{n} k_i u_i(\tilde{X}_i).$$

Hence

$$\sum_{i=1}^{n} k_i \mathrm{E}[u_i(X_i)] \leq \sum_{i=1}^{n} k_i \mathrm{E}[u_i(\tilde{X}_i)].$$

This shows that expression (63) is indeed maximal for $(\tilde{X}_1, \ldots, \tilde{X}_n)$. □

Example 7

Suppose that all companies use an exponential utility function,

$$u_j(x) = \frac{1}{a_j}[1 - \exp(-a_j x)],$$

where a_j is the constant risk aversion of company j, $j = 1, \ldots, n$. From (67), we get

$$k_j \exp(-a_j \tilde{X}_j) = \Lambda \quad (69)$$

or

$$\tilde{X}_j = -\frac{\ln \Lambda}{a_j} + \frac{\ln k_j}{a_j}. \quad (70)$$

Summing over j, we obtain an equation that determines Λ:

$$W = -\sum_{j=1}^{n} \frac{1}{a_j} \ln \Lambda + \sum_{j=1}^{n} \frac{\ln k_j}{a_j}. \quad (71)$$

Let us introduce a, which is defined by the equation

$$\frac{1}{a} = \frac{1}{a_1} + \ldots + \frac{1}{a_n}. \quad (72)$$

Then it follows from (71) that

$$-\ln \Lambda = aW - a \sum_{j=1}^{n} \frac{\ln k_j}{a_j}. \quad (73)$$

Substitution in (70) yields

$$\tilde{X}_i = \frac{a}{a_i} W + \frac{\ln k_i}{a_i} - \frac{a}{a_i} \sum_{j=1}^{n} \frac{\ln k_j}{a_j} \quad (74)$$

for $i = 1, \ldots, n$. Thus company i will assume the fraction (or quota) $q_i = a/a_i$ of total wealth W plus a possibly negative side payment

$$d_i = \frac{\ln k_i}{a_i} - \frac{a}{a_i} \sum_{j=1}^{n} \frac{\ln k_j}{a_j}. \quad (75)$$

It is easily verified that

$$q_1 + \ldots + q_n = 1 \quad (76)$$

and

$$d_1 + \ldots + d_n = 0. \quad (77)$$

We note that the q_i's are inversely proportional to the risk aversions and that they are the same for all Pareto optimal risk exchanges. Pareto optimal risk exchanges differ only by their side payments.

Example 8

Suppose now that all companies use a power utility function of the first kind, such that

$$u_j(x) = \frac{s_j^{c+1} - (s_j - x)^{c+1}}{(c+1)s_j^c}, \qquad j = 1, \ldots, n, \quad (78)$$

where s_j is the level of saturation of company j. From (67) we get

$$k_j \left(1 - \frac{\tilde{X}_j}{s_j}\right)^c = \Lambda \quad (79)$$

or

$$\tilde{X}_j = -\frac{s_j}{k_j^{1/c}} \Lambda^{1/c} + s_j. \quad (80)$$

Summing over j, we obtain an equation which determines Λ:

$$W = -\sum_{j=1}^{n} \frac{s_j}{k_j^{1/c}} \Lambda^{1/c} + \sum_{j=1}^{n} s_j. \quad (81)$$

Let

$$s = s_1 + \ldots + s_n \quad (82)$$

denote the combined level of saturation. Then it follows from (81) that

$$\Lambda^{1/c} = \frac{s - W}{\sum_{j=1}^{n} \frac{s_j}{k_j^{1/c}}}. \tag{83}$$

Substitution in (80) yields

$$\tilde{X}_i = \frac{\frac{s_i}{k_i^{1/c}}}{\sum_{j=1}^{n} \frac{s_j}{k_j^{1/c}}} W + s_i - \frac{\frac{s_i}{k_i^{1/c}}}{\sum_{j=1}^{n} \frac{s_j}{k_j^{1/c}}} s \tag{84}$$

for $i = 1, \ldots, n$. Hence again $\tilde{X}_i$ is of the form

$$\tilde{X}_i = q_i W + d_i. \tag{85}$$

But note that now both the quotas and the side payments vary, such that

$$d_i = s_i - q_i s. \tag{86}$$

If we write this result in the form

$$s_i - \tilde{X}_i = q_i(s - W), \tag{87}$$

it has the following interpretation: The expression $s_i - \tilde{X}_i$ is the amount that is missing for maximal satisfaction. It is a fixed percentage of $s - W$, which is the total amount missing for all companies combined.

Example 9

Consider n investors with identical power utility functions of the second kind

$$u_j(x) = \frac{x^{1-c} - 1}{1 - c}, \qquad j = 1, \ldots, n.$$

From (67), we see that

$$k_j \tilde{X}_j^{-c} = \Lambda \tag{88}$$

or

$$\tilde{X}_j = k_j^{1/c} \Lambda^{-1/c}. \tag{89}$$

Summing over j, we get

$$W = \sum_{j=1}^{n} k_j^{1/c} \Lambda^{-1/c}, \tag{90}$$

or

$$\Lambda^{-1/c} = \frac{W}{\sum_{j=1}^{n} k_j^{1/c}}. \tag{91}$$

If we substitute this in (89) we see that

$$\tilde{X}_i = q_i W \tag{92}$$

with

$$q_i = \frac{k_i^{1/c}}{\sum_{j=1}^{n} k_j^{1/c}}. \tag{93}$$

Hence each investor assumes a fixed quota of total wealth. As in the case of power utility functions of the first kind, the quotas vary, but now there are no side payments.

Example 10

Let $n = 2$. Suppose that $u_1(x) = x$ and $u_2(x) = u(x)$, a utility function in the proper sense with $u''(x) < 0$. Then condition (67) tells us that

$$k_1 = k_2 u'(\tilde{X}_2).$$

But this means that $\tilde{X}_2$ is a constant, say d. Hence $\tilde{X}_1 = W - d$. This result is not really surprising: since company 1 is not risk averse, it will assume all the risk!

We have presented selected examples in which the Pareto optimal risk exchanges are of an attractively simple form. In general, this is not the case. The following example illustrates the point.

Example 11

Let $n = 2$. Suppose that $u_1(x)$ and $u_2(x)$ are power utility functions of the second kind with parameters $c_1 = 1$ and $c_2 = 2$, that is, that

$$u_1(x) = \ln x,\ u_2(x) = 1 - \frac{1}{x} \qquad \text{for } x > 0.$$

From (67) we obtain the condition that

$$\frac{1}{k_1} \tilde{X}_1 = \frac{1}{k_2} (\tilde{X}_2)^2. \tag{94}$$

Together with the condition that $\tilde{X}_1 + \tilde{X}_2 = W$, this results in a quadratic equation. Its solution is

$$\tilde{X}_1 = W - \frac{1}{2}\left(\sqrt{a^2 + 4aW} - a\right), \tag{95}$$

$$\tilde{X}_2 = \frac{1}{2}\left(\sqrt{a^2 + 4aW} - a\right), \tag{96}$$

with $a = k_2/k_1$. Here $\tilde{X}_1$ and $\tilde{X}_2$ are obviously not linear functions of W.

Example 7 helps us to understand the Pareto optimal risk exchanges in the general case. Let $u_1(x)$, . . . , $u_n(x)$ be arbitrary utility functions, and let $(\tilde{X}_1, \ldots, \tilde{X}_n)$ be a Pareto optimal risk exchange. For given $W = w$, $(\tilde{X}_1, \ldots, \tilde{X}_n)$ maximizes expression (62). Hence $\tilde{X}_i = \tilde{X}_i(w)$ is a function of the total wealth w. Let $j \neq h$. According to Theorem 1

$$k_j u_j'(\tilde{X}_j(w)) = k_h u_h'(\tilde{X}_h(w)). \tag{97}$$

Differentiation with respect to w yields

$$k_j u_j''(\tilde{X}_j(w)) \frac{d\tilde{X}_j}{dw} = k_h u_h''(\tilde{X}_h(w)) \frac{d\tilde{X}_h}{dw}. \tag{98}$$

Dividing (98) by (97) we see that

$$r_j(\tilde{X}_j(w)) \frac{d\tilde{X}_j}{dw} = r_h(\tilde{X}_h(w)) \frac{d\tilde{X}_h}{dw}. \tag{99}$$

From this and the observation that

$$d\tilde{X}_1 + \ldots + d\tilde{X}_n = dw, \tag{100}$$

it follows that

$$d\tilde{X}_j = \frac{\dfrac{1}{r_j(\tilde{X}_j)}}{\sum_{h=1}^{n} \dfrac{1}{r_h(\tilde{X}_h)}} dw, \qquad j = 1, \ldots, n. \tag{101}$$

Thus the family of Pareto optimal risk exchanges can be obtained as follows. For a particular value of w, say w_0, we can choose $\tilde{X}_1(w_0), \ldots, \tilde{X}_n(w_0)$. Then $\tilde{X}_1(w), \ldots, \tilde{X}_n(w)$ are determined as the solution of (101), subject to the boundary condition at $w = w_0$.

As an application of (101), we revisit Examples 8 and 9. For a unified treatment, we suppose that

$$\frac{1}{r_j(x)} = \alpha x + \beta_j, \qquad j = 1, \ldots, n. \tag{102}$$

We want to verify that a Pareto optimal risk exchange is of the form

$$\tilde{X}_j = q_j W + d_j, \tag{103}$$

or equivalently,

$$d\tilde{X}_j = q_j\, dw \tag{104}$$

for a set of quotas $q_1, \ldots, q_n$ and side payments $d_1, \ldots, d_n$. From (101) and (102) we obtain

$$d\tilde{X}_j = \frac{\alpha \tilde{X}_j + \beta_j}{\sum_{h=1}^{n} (\alpha \tilde{X}_h + \beta_h)} dw = \frac{\alpha \tilde{X}_j + \beta_j}{\alpha W + \beta} dw \tag{105}$$

with $\beta = \beta_1 + \ldots + \beta_n$. Hence, by (103)

$$d\tilde{X}_j = \frac{\alpha q_j W + \alpha d_j + \beta_j}{\alpha W + \beta} dw \tag{106}$$

To see when this ratio is equal to q_j, we distinguish two cases:

(1) If $\beta \neq 0$, it suffices to set

$$q_j = \frac{\alpha d_j + \beta_j}{\beta}, \qquad j = 1, \ldots, n.$$

(2) If $\beta = 0$, $q_1, \ldots, q_n$ are arbitrary quotas, and the side payments are fixed:

$$d_j = -\frac{\beta_j}{\alpha}, \qquad j = 1, \ldots, n.$$

Theorem 1 tells us that for a Pareto optimal risk exchange $(\tilde{X}_1, \ldots, \tilde{X}_n)$, there is a random variable Λ such that

$$\Lambda = k_i u_i'(\tilde{X}_i), \qquad \text{for } i = 1, \ldots, n. \tag{107}$$

Since $\tilde{X}_i = \tilde{X}_i(w)$ is a function of total wealth w, it follows that $\Lambda = \Lambda(w)$ is a function of w. Differentiating (107), we get

$$\Lambda' = k_i u_i''(\tilde{X}_i)\tilde{X}_i'. \tag{108}$$

Dividing this equation by (107) and using (101), we obtain

$$\frac{\Lambda'}{\Lambda} = -\frac{1}{\sum_{h=1}^{n} \dfrac{1}{r_h(\tilde{X}_h)}}. \tag{109}$$

This shows that Λ is a decreasing function of total wealth.

9. THE SYNERGY POTENTIAL

We consider the n companies introduced in the preceding section and assume that $(W_1, \ldots, W_n)$, the allocation of their total wealth W, is not Pareto optimal. How much can the companies gain through cooperation?

An answer is provided by the *synergy potential* η. This is the largest amount x that can be extracted from the system without hurting any of the companies, that is, such that there is a risk exchange $(X_1, \ldots, X_n)$ with

$$X_1 + \ldots + X_n = W - x \tag{110}$$

and

$$\mathrm{E}[u_i(X_i)] \geq \mathrm{E}[u_i(W_i)], \qquad \text{for } i = 1, \ldots, n. \tag{111}$$

It is clear that for $x = \eta$ we must have equality in (111) and $(\tilde{X}_1, \ldots, \tilde{X}_n)$ must be a Pareto optimal risk exchange of $W - \eta$.

Example 12 (continued from Example 7)

Suppose that all utility functions are exponential. Since $(\tilde{X}_1, \ldots, \tilde{X}_n)$ is a Pareto optimal risk exchange of $W - \eta$, it follows that

$$\tilde{X}_i = \frac{a}{a_i}(W - \eta) + d_i, \qquad \text{for } i = 1, \ldots, n. \quad (112)$$

Then we use the condition that

$$\mathrm{E}[u_i(\tilde{X}_i)] = \mathrm{E}[u_i(W_i)] \quad (113)$$

to see that

$$\mathrm{E}[e^{-aW}]e^{a\eta - a_i d_i} = \mathrm{E}[e^{-a_i W_i}], \quad (114)$$

or

$$\frac{a}{a_i}\eta - d_i = \frac{1}{a_i}\ln \mathrm{E}[e^{-a_i W_i}] - \frac{1}{a_i}\ln \mathrm{E}[e^{-aW}]. \quad (115)$$

Summation over i yields an explicit expression for the synergy potential:

$$\eta = \sum_{i=1}^{n} \frac{1}{a_i}\ln \mathrm{E}[e^{-a_i W_i}] - \frac{1}{a}\ln E[e^{-aW}]$$

$$= \ln \frac{\prod_{i=1}^{n} \mathrm{E}[e^{-a_i W_i}]^{1/a_i}}{\mathrm{E}[e^{-aW}]^{1/a}}. \quad (116)$$

Example 13 (continued from Example 8)

We assume that the companies use power utility functions of the first kind. According to (87)

$$s_i - \tilde{X}_i = q_i(s - W + \eta). \quad (117)$$

From

$$\mathrm{E}[u_i(\tilde{X}_i)] = \mathrm{E}[u_i(W_i)]$$

it follows that

$$q_i^{c+1}\,\mathrm{E}[(s - W + \eta)^{c+1}] = \mathrm{E}[(s_i - W_i)^{c+1}]. \quad (118)$$

Taking the $(c + 1)$-th root and summing over i, we get

$$\mathrm{E}[(s - W + \eta)^{c+1}]^{1/(c+1)} = \sum_{i=1}^{n} \mathrm{E}[(s_i - W_i)^{c+1}]^{1/(c+1)}. \quad (119)$$

This is an implicit equation for the synergy potential η.

Example 14 (continued from Example 9)

Suppose that each of n investors has a power utility function of the second kind. Hence

$$\tilde{X}_i = q_i(W - \eta). \quad (120)$$

From

$$\mathrm{E}[u_i(\tilde{X}_i)] = \mathrm{E}[u_i(W_i)]$$

we get

$$q_i^{1-c}\,\mathrm{E}[(W - \eta)^{1-c}] = \mathrm{E}[W_i^{1-c}] \qquad \text{if } c \neq 1, \quad (121)$$

and

$$\ln q_i + \mathrm{E}[\ln(W - \eta)] = \mathrm{E}[\ln W_i] \quad \text{if } c = 1. \quad (122)$$

Taking the $(1 - c)$-th root in (121) and summing over i, we obtain the equation

$$\mathrm{E}[(W - \eta)^{1-c}]^{1/(1-c)} = \sum_{i=1}^{n} \mathrm{E}[W_i^{1-c}]^{1/(1-c)}, \quad (123)$$

which determines η if $c \neq 1$. By exponentiating (122) and summing over i we obtain the equation

$$e^{\mathrm{E}[\ln(W-\eta)]} = \sum_{i=1}^{n} e^{\mathrm{E}[\ln W_i]}, \quad (124)$$

which determines η if $c = 1$.

Example 15

In the situation of Example 10, equality of the expected utilities implies that

$$\tilde{X}_1 = W - \mathrm{E}[W_2] \quad (125)$$

and $\tilde{X}_2 = d$, where

$$u(d) = \mathrm{E}[u(W_2)]. \quad (126)$$

Thus $d = \pi$, the certainty equivalent of W_2. It follows that

$$\eta = W - (\tilde{X}_1 + \pi)$$

$$= \mathrm{E}[W_2] - \pi. \quad (127)$$

We can use the synergy potential to construct a particular Pareto optimal risk exchange. The idea is to first extract η from the companies and then to distribute η to the companies according to (101). The resulting Pareto optimal risk exchange $(\tilde{X}_1, \ldots, \tilde{X}_n)$ of W is characterized by the condition that

$$\mathrm{E}[u_i(\tilde{X}_i(W - \eta))] = \mathrm{E}[u_i(W_i)], \qquad \text{for } i = 1, \ldots, n. \quad (128)$$

Example 16 (continued from Example 12)

In the case of exponential utility functions we have

$$\tilde{X}_i = \frac{a}{a_i} W + d_i.$$

To determine the side payments, we substitute this expression in (128) to see that

$$\mathrm{E}[e^{-aW}]\, e^{a\eta - a_i d_i} = \mathrm{E}[e^{-a_i W_i}].$$

From this it follows that

$$d_i = \frac{a}{a_i}\eta + \frac{1}{a_i}\ln \mathrm{E}[e^{-aW}] - \frac{1}{a_i}\ln \mathrm{E}[e^{-a_i W_i}]. \quad (129)$$

Substituting for η, we obtain finally the result that

$$d_i = \frac{a}{a_i}\sum_{j=1}^{n}\frac{1}{a_j}\ln \mathrm{E}[e^{-a_j W_j}] - \frac{1}{a_i}\ln \mathrm{E}[e^{-a_i W_i}],$$

$$\text{for } i = 1, \ldots, n. \quad (130)$$

Example 17

For power utility functions of the first kind, we found that a Pareto optimal risk exchange $(\tilde{X}_1, \ldots, \tilde{X}_n)$ is such that

$$s_i - \tilde{X}_i = q_i(s - W).$$

From this and (128), we obtain the condition that

$$q_i^{c+1}\mathrm{E}[(s - W + \eta)^{c+1}] = \mathrm{E}[(s_i - W_i)^{c+1}].$$

Thus

$$q_i = \left(\frac{\mathrm{E}[(s_i - W_i)^{c+1}]}{\mathrm{E}[(s - W + \eta)^{c+1}]}\right)^{1/(c+1)}. \quad (131)$$

Finally, we use (119) to get an explicit formula for the resulting quota:

$$q_i = \frac{\mathrm{E}[(s_i - W_i)^{c+1}]^{1/(c+1)}}{\sum_{j=1}^{n}\mathrm{E}[(s_j - W_j)^{c+1}]^{1/(c+1)}}, \qquad \text{for } i = 1, \ldots, n.$$

$$(132)$$

Example 18

For power utility functions of the second kind, we found that $\tilde{X}_i = q_i W$. From condition (128), we see that

$$q_i = \left(\frac{\mathrm{E}[W_i^{1-c}]}{\mathrm{E}[(W - \eta)^{1-c}]}\right)^{1/(1-c)}, \qquad \text{if } c \neq 1, \quad (133)$$

and

$$q_i = \frac{e^{\mathrm{E}[\ln W_i]}}{e^{\mathrm{E}[\ln(W-\eta)]}}, \qquad \text{if } c = 1. \quad (134)$$

From (123) and (124), it follows that

$$q_i = \frac{\mathrm{E}[W_i^{1-c}]^{1/(1-c)}}{\sum_{j=1}^{n}\mathrm{E}[W_j^{1-c}]^{1/(1-c)}}, \qquad \text{if } c \neq 1, \quad (135)$$

and

$$q_i = \frac{e^{\mathrm{E}[\ln W_i]}}{\sum_{j=1}^{n} e^{\mathrm{E}[\ln W_j]}}, \qquad \text{if } c = 1. \quad (136)$$

In the Appendix we derive Hölder's inequality and Minkowski's inequality as a by-product of Examples 12 and 14.

10. MARKET AND EQUILIBRIUM

Again we consider the n companies that were introduced in Section 8. We concluded that the companies should settle on a Pareto optimal risk exchange. Because this is a rich family, more definite answers are desirable. In the last section we proposed a particular Pareto optimal risk exchange. In this section an alternative proposal, due to Borch and Bühlmann, is discussed, which is based on economic ideas.

We suppose that random payments are traded in a market, whereby the price $H(Y)$ for any payment Y (a random variable) is calculated as

$$H(Y) = \mathrm{E}[\Psi Y]. \quad (137)$$

Here Ψ is a positive random variable. We assume that $H(Y)$ represents the price as of the end of the year. Hence the price of a constant payment must be identical to this constant. Therefore we must have $\mathrm{E}[\Psi] = 1$. By writing the right-hand side of (137) as $\mathrm{E}[Y] + \mathrm{E}[\Psi Y] - \mathrm{E}[\Psi]\mathrm{E}[Y]$, we see that the price of Y can also be written in the form

$$H(Y) = \mathrm{E}[Y] + \mathrm{Cov}(Y, \Psi), \quad (138)$$

that is, the price of a payment is its expectation modified by an adjustment that takes into account the market conditions. Alternatively, we can interpret the price of a payment as its expectation with respect to a modified probability measure, Q, that is defined by the relation

$$\mathrm{E}_Q[Y] = \mathrm{E}[\Psi Y] \qquad \text{for all } Y. \quad (139)$$

In other words, Ψ is the Radon-Nikodym derivative of the Q-measure with respect to the original probability measure. For this reason Bühlmann (1980, 1984) calls Ψ a *price density*.

Company i will want to buy a payment Y_i in order to

$$\text{maximize} \qquad E[u_i(W_i + Y_i - H(Y_i))]. \tag{140}$$

A payment $\tilde{Y}_i$ solves this problem if and only if the condition

$$u_i'(W_i + \tilde{Y}_i - H(\tilde{Y}_i)) = \Psi E[u_i'(W_i + \tilde{Y}_i - H(\tilde{Y}_i))] \tag{141}$$

is satisfied.

To see the necessity of this condition, suppose that $\tilde{Y}_i$ is a solution of (140). Let V be an arbitrary random variable; we consider the family

$$Y_i = \tilde{Y}_i + tV.$$

According to our assumption, the function

$$f(t) = E[u_i(W_i + Y_i - H(Y_i))]$$

is maximal for $t = 0$. Hence

$$f'(0) = E[u_i'(W_i + \tilde{Y}_i - H(\tilde{Y}_i))(V - E[\Psi V])] = 0. \tag{142}$$

We rewrite this equation as

$$E[V\{u_i'(W_i + \tilde{Y}_i - H(\tilde{Y}_i)) - \Psi E[u_i'(W_i + \tilde{Y}_i - H(\tilde{Y}_i))]\}] = 0. \tag{143}$$

Since it is valid for all V, the random variable inside the braces must be zero, and condition (141) follows.

To see that condition (141) is sufficient, consider a payment $\tilde{Y}$ that satisfies (141) and any other payment Y. From (57) it follows that

$$\begin{aligned} u_i(W_i+Y_i-H(Y_i)) &\le u_i(W_i+\tilde{Y}_i-H(\tilde{Y}_i)) \\ &\quad + u_i'(W_i+\tilde{Y}_i-\mathrm{H}(\tilde{Y}_i))(Y_i-H(Y_i)-\tilde{Y}_i+H(\tilde{Y}_i)) \\ &= u_i(W_i+\tilde{Y}_i-H(\tilde{Y}_i)) \\ &\quad + \Psi E[u_i'(W_i+\tilde{Y}_i-H(\tilde{Y}_i))](Y_i-H(Y_i)-\tilde{Y}_i+H(\tilde{Y}_i)). \end{aligned} \tag{144}$$

Taking expectations and using the definition of H, we see that

$$E[u_i(W_i + Y_i - H(Y_i))] \le E[u_i(W_i + \tilde{Y}_i - H(\tilde{Y}_i))], \tag{145}$$

which completes the proof.

We note that the optimal $\tilde{Y}_i$ is unique apart from an additive constant; hence $\tilde{Y}_i - H(\tilde{Y}_i)$ is unique. It can be interpreted as the optimal payment that has a zero price, and we refer to it as the *net demand* of company i.

Given Ψ, the random variable

$$\sum_{i=1}^{n} [\tilde{Y}_i - H(\tilde{Y}_i)] \tag{146}$$

is the *excess demand.* The companies can maximize simultaneously their expected utilities only if the excess demand vanishes (this is the *market clearing condition*). This leads us to the following definition.

A price density Ψ and the payments $\tilde{Y}_1, \ldots, \tilde{Y}_n$ constitute an *equilibrium,* if (146) vanishes and if (141) is satisfied for $i = 1, \ldots, n$.

Note that an equilibrium induces a risk exchange $(\tilde{X}_1, \ldots, \tilde{X}_n)$, with

$$\tilde{X}_i = W_i + \tilde{Y}_i - H(\tilde{Y}_i), \qquad \text{for } i = 1, \ldots, n. \tag{147}$$

Then condition (141) states that

$$u_i'(\tilde{X}_i) = \Psi E[u_i'(\tilde{X}_i)], \qquad \text{for } i = 1, \ldots, n. \tag{148}$$

From this and Theorem 1 it follows that the risk exchange implied by an equilibrium is Pareto optimal. Furthermore, (109) is satisfied with $\Lambda = \Psi$. In particular, this shows that Ψ is a decreasing function of total wealth.

The converse is true in the following sense. Suppose that $(W_1, \ldots, W_n)$ is already Pareto optimal; then $W_1, \ldots, W_n$ and Ψ constitute an equilibrium, if we set

$$\Psi = \frac{u_i'(W_i)}{E[u_i'(W_i)]}. \tag{149}$$

Moreover,

$$\tilde{Y}_i - H(\tilde{Y}_i) = 0 \qquad \text{for } i = 1, \ldots, n.$$

This can be seen from (67) (with $\tilde{X}_i$ replaced by W_i) and (141).

Example 19 (continued from Example 7)

Assuming that all companies use exponential utility functions, we gather from (141) that

$$\tilde{Y}_i = -W_i - \frac{1}{a_i} \ln \Psi + \kappa_i, \tag{150}$$

where κ_i is a constant. Hence the net demand of company i is

$$\tilde{Y}_i - H(\tilde{Y}_i) = -W_i - \frac{1}{a_i}\ln\Psi + \mathrm{E}[\Psi W_i] + \frac{1}{a_i}\mathrm{E}[\Psi\ln\Psi]. \quad (151)$$

In the equilibrium the sum over i must vanish. Hence

$$0 = -W - \frac{1}{a}\ln\Psi + \kappa, \quad (152)$$

where κ is a constant. Since $\mathrm{E}[\Psi] = 1$, it follows that the equilibrium price density is

$$\Psi = \frac{e^{-aW}}{\mathrm{E}[e^{-aW}]}. \quad (153)$$

Finally, a little calculation shows that

$$\begin{aligned}\tilde{X}_i &= W_i + \tilde{Y}_i - H(\tilde{Y}_i)\\ &= \frac{a}{a_i}W + \mathrm{E}[\Psi W_i] - \frac{a}{a_i}\mathrm{E}[\Psi W]\\ &= \frac{a}{a_i}W + H(W_i) - \frac{a}{a_i}H(W) \quad (154)\end{aligned}$$

in the equilibrium.

Example 20 (continued from Example 8)

We assume that the companies use power utility functions of the first kind. Hence

$$u_i'(x) = \frac{(s_i - x)^c}{s_i^c}, \qquad i = 1, \ldots, n,$$

and

$$s_i - \tilde{X}_i = q_i(s - W), \qquad i = 1, \ldots, n;$$

see (87). Then according to (148) the equilibrium price density is

$$\Psi = \frac{u_i'(\tilde{X}_i)}{\mathrm{E}[u_i'(\tilde{X}_i)]} = \frac{(s - W)^c}{\mathrm{E}[(s - W)^c]}. \quad (155)$$

The equilibrium quotas are best determined from the condition that $H(W_i) = H(\tilde{X}_i)$, or

$$\begin{aligned}H(W_i) &= H(s_i - q_i(s - W))\\ &= s_i - q_i s + q_i H(W). \quad (156)\end{aligned}$$

Hence

$$q_i = \frac{s_i - H(W_i)}{s - H(W)} = \frac{\mathrm{E}[\Psi(s_i - W_i)]}{\mathrm{E}[\Psi(s - W)]}, \quad \text{for } i = 1, \ldots, n. \quad (157)$$

Example 21 (continued from Example 9)

If all companies use the same power utility function of the second kind,

$$u_i'(x) = x^{-c}, \qquad i = 1, \ldots, n,$$

we know that

$$\tilde{X}_i = q_i W, \qquad i = 1, \ldots, n;$$

see (92). Hence the equilibrium price density is

$$\Psi = \frac{u_i'(\tilde{X}_i)}{\mathrm{E}[u_i'(\tilde{X}_i)]} = \frac{W^{-c}}{\mathrm{E}[W^{-c}]}. \quad (158)$$

Again, the equilibrium quotas are best obtained from the condition that $H(W_i) = H(\tilde{X}_i) = q_i H(W)$. Thus

$$q_i = \frac{H(W_i)}{H(W)} = \frac{\mathrm{E}[\Psi W_i]}{\mathrm{E}[\Psi W]} = \frac{\mathrm{E}[W^{-c}W_i]}{\mathrm{E}[W^{-c+1}]}, \quad \text{for } i = 1, \ldots, n. \quad (159)$$

Remark 4

From (138) it follows that for any random variable Y

$$H(Y) - E[Y] = \beta(H(W) - \mathrm{E}(W)) \quad (160)$$

with

$$\beta = \frac{\mathrm{Cov}(Y, \Psi)}{\mathrm{Cov}(W, \Psi)}, \quad (161)$$

where Ψ is now the equilibrium price density. Formula (160) is close to a central result in the capital-asset-pricing model (CAPM). As an illustration, we revisit our three preceding examples. Thus

$$\beta = \frac{\mathrm{Cov}(Y, e^{-aW})}{\mathrm{Cov}(W, e^{-aW})} \quad (162)$$

in Example 19,

$$\beta = \frac{\mathrm{Cov}(Y, (s - W)^c)}{\mathrm{Cov}(W, (s - W)^c)} \quad (163)$$

in Example 20, and

$$\beta = \frac{\mathrm{Cov}(Y, W^{-c})}{\mathrm{Cov}(W, W^{-c})} \quad (164)$$

in Example 21. Note that for $c = 1$ (quadratic utility functions), (163) reduces to the classical CAPM formula

$$\beta = \frac{\mathrm{Cov}(Y, W)}{\mathrm{Var}[W]}. \quad (165)$$

11. PRICING OF DERIVATIVE SECURITIES

In the equilibrium the price of a payment Y is $H(Y)$, given by formulas (137), (138) or (139), where Ψ is the equilibrium price density. Typically, the random variable Y is the value of an *asset* or a *derivative security* at the end of a period. Under certain assumptions, the price of a derivative security can be expressed in terms of the price of the underlying asset.

First, we assume that the random variable Ψ has a lognormal distribution, that is,

$$\Psi = e^{Z}, \tag{166}$$

where Z has a normal distribution, say with variance ν^2. Since

$$\mathrm{E}[\Psi] = \exp\left(\mathrm{E}[Z] + \frac{1}{2}\nu^2\right) \tag{167}$$

must be 1, it follows that $\mathrm{E}[Z] = -(1/2)\nu^2$. According to Formulas (153), (155), and (158), the assumption of lognormality for Ψ means that W is normal in Example 19, that $s - W$ is lognormal in Example 20, or that W is lognormal in Example 21.

Let us consider a particular asset. We denote its value at the end of the period by S and assume that the random variable S has a lognormal distribution. Then we can write

$$S = s_0\, e^{R}, \tag{168}$$

where s_0 is the observed price of the asset at the beginning of the period, and R has a normal distribution, say, with mean μ and variance σ^2. We assume that the joint distribution of (Z, R) is bivariate normal with coefficient of correlation ρ. Then we obtain the following expression for the moment-generating function of R with respect to the Q-measure:

$$\mathrm{E}_Q[e^{tR}] = \mathrm{E}[\Psi e^{tR}] = \mathrm{E}[e^{Z+tR}] = \exp\left[t(\mu + \rho\nu\sigma) + \frac{1}{2}t^2\sigma^2\right]. \tag{169}$$

This shows that in the Q-measure the distribution of R is still normal, with unchanged variance σ^2 and new mean

$$\mu_Q = \mu + \rho\nu\sigma. \tag{170}$$

Luckily, there is a more practical expression for μ_Q. Since s_0 is the price of the asset at the *beginning of the period,* we have

$$s_0 = e^{-\delta}\, H(S) = e^{-\delta}\, \mathrm{E}_Q[S], \tag{171}$$

where δ is the risk-free force of interest. Hence we obtain the equation

$$s_0 = e^{-\delta}\, s_0\, \mathrm{E}_Q[e^{R}] = e^{-\delta}\, s_0 \exp\left(\mu_Q + \frac{1}{2}\sigma^2\right), \tag{172}$$

which yields

$$\mu_Q = \delta - \frac{1}{2}\sigma^2. \tag{173}$$

Now let us consider a derivative security, whose value at the end of the period is $f(S)$, a function of the underlying asset. Its price at the beginning of the period is

$$e^{-\delta}\, H(f(S)) = e^{-\delta}\, \mathrm{E}_Q[f(s_0\, e^{R})], \tag{174}$$

where R is normal with mean given by (173) and variance σ^2. For example, for a European call option with strike price K, $f(S) = (S - K)_+$. Then (174) can be calculated explicitly, which leads to the Black-Scholes formula.

Remark 5

The method can be generalized to price derivative securities that depend on several, say, m assets. Let

$$S_i = s_{i0}\, e^{R_i}, \tag{175}$$

denote the value at the end of the period of asset i, where s_{i0} is the observed price of asset i at the beginning of the period, $i = 1, \ldots, m$. The assumption is now that $(Z, R_1, \ldots, R_m)$ has a multivariate normal distribution. Then in the Q-measure $(R_1, \ldots, R_m)$ has still a multivariate normal distribution, with unchanged covariance matrix, but modified mean vector, such that

$$\mathrm{E}_Q[R_i] = \delta - \frac{1}{2}\mathrm{Var}[R_i], \text{ for } i = 1, \ldots, m. \tag{176}$$

In the framework of Examples 19–21, practical results can also be obtained for derivative securities on assets for which S is a linear function of W.

In Example 19 suppose that $S = qW$. Then

$$\mathrm{E}_Q[S] = \frac{\mathrm{E}[Se^{-aW}]}{\mathrm{E}[e^{-aW}]} = \frac{\mathrm{E}[Se^{-\alpha S}]}{\mathrm{E}[e^{-\alpha S}]} \tag{177}$$

with $\alpha = a/q$. According to (171), the value of α is determined from the condition that

$$\frac{\mathrm{E}[Se^{-\alpha S}]}{\mathrm{E}[e^{-\alpha S}]} = e^{\delta}\, s_0. \tag{178}$$

Then the price of a derivative security with payoff $f(S)$ is given by the expression

$$e^{-\delta}\, \mathrm{E}_Q[f(S)] = e^{-\delta} \frac{\mathrm{E}[f(S)e^{-\alpha S}]}{\mathrm{E}[e^{-\alpha S}]}. \qquad (179)$$

This is the Esscher method in the sense of Bühlmann. We note that it also works for assets where S and $W - S$ are independent random variables: here

$$\mathrm{E}_Q[S] = \frac{\mathrm{E}[Se^{-aS}\, e^{-a(W-S)}]}{\mathrm{E}[e^{-aS}\, e^{-a(W-S)}]}$$
$$= \frac{\mathrm{E}[Se^{-aS}]\mathrm{E}[e^{-a(W-S)}]}{\mathrm{E}[e^{-aS}]\mathrm{E}[e^{-a(W-S)}]} = \frac{\mathrm{E}[Se^{-aS}]}{\mathrm{E}[e^{-aS}]}. \qquad (180)$$

Hence a is determined from (178) with α replaced by a.

In Example 20 we suppose that $S = q(s - W)$. Then

$$\mathrm{E}_Q[S] = \frac{\mathrm{E}[S(s-W)^c]}{\mathrm{E}[(s - W)^c]} = \frac{\mathrm{E}[SS^c]}{\mathrm{E}[S^c]}. \qquad (181)$$

The value of c is determined from the condition that

$$\frac{\mathrm{E}[S^{1+c}]}{\mathrm{E}[S^c]} = e^{\delta}\, s_0, \qquad (182)$$

and the price of a derivative security with payoff $f(S)$ is given by the expression

$$e^{-\delta}\, \mathrm{E}_Q[f(S)] = e^{-\delta} \frac{\mathrm{E}[f(S)S^c]}{\mathrm{E}[S^c]}. \qquad (183)$$

In Example 21 we suppose again $S = qW$. Then

$$\mathrm{E}_Q[S] = \frac{\mathrm{E}[SW^{-c}]}{\mathrm{E}[W^{-c}]} = \frac{\mathrm{E}[SS^{-c}]}{\mathrm{E}[S^{-c}]}. \qquad (184)$$

The value of c is now determined from the condition that

$$\frac{\mathrm{E}[S^{1-c}]}{\mathrm{E}[S^{-c}]} = e^{\delta}\, s_0, \qquad (185)$$

and the price of a derivative security with payoff $f(S)$ is given by the expression

$$e^{-\delta}\, \mathrm{E}_Q[f(S)] = e^{-\delta} \frac{\mathrm{E}[f(S)S^{-c}]}{\mathrm{E}[S^{-c}]}. \qquad (186)$$

Formula (183) is also acceptable, if c, the solution of (182) is negative. In this case the solution of (185) is positive, which leads to (186). But this is again (183), with a negative c. Formulas (182) and (183) summarize the Esscher method that was proposed by Gerber and Shiu (1994a, 1994b).

Remark 6

Using (178), we can rewrite (179) as

$$e^{-\delta}\, \mathrm{E}_Q[f(S)] = s_0 \frac{\mathrm{E}[f(S)e^{-\alpha S}]}{\mathrm{E}[Se^{-\alpha S}]}. \qquad (187)$$

Similarly, (183) can be rewritten as

$$e^{-\delta}\, \mathrm{E}_Q[f(S)] = s_0 \frac{\mathrm{E}[f(S)S^c]}{\mathrm{E}[S^{1+c}]}. \qquad (188)$$

It may be surprising that δ does not appear in these expressions for the prices, but of course the values of α and c are functions of δ.

Remark 7

The Esscher method summarized by Formulas (182) and (183) has some attractive features. For example, if S has a lognormal distribution, it has also a lognormal distribution in the Q-measure. In particular, it reproduces the formula of Black-Scholes.

12. BIBLIOGRAPHICAL NOTES

A broad, less self-contained review has been given by Aase (1993). The article by Taylor (1992a) is highly recommended.

In Section 5 the premiums are determined by the principle of equivalent utility. If this principle is adopted in a dynamic model, there is an intrinsic relationship between the underlying utility function and the resulting probability of ruin; see Gerber (1975).

The optimality of a stop-loss contract of Section 6 seems to have been discovered by Arrow (1963). Its minimal variance property has been discussed by others, for example, by Kahn (1961).

The theorem of Borch in Section 8 can be found in the books of Bühlmann (1970) and Gerber (1979). In some of the literature, the family of utility functions satisfying (102) is called the HARA family (hyperbolic absolute risk aversion).

In Sections 9 and 10 we discussed Pareto optimal risk exchanges of a specific form. Other proposals have been discussed by Bühlmann and Jewell (1979) and by Baton and Lemaire (1981). Solutions that are not Pareto optimal have been proposed by Chan and Gerber (1985), Gerber (1984), and Taylor (1992b).

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APPENDIX

ECONOMIC PROOFS OF TWO FAMOUS INEQUALITIES

In Section 9 we introduced the synergy potential. By observing that this quantity is non-negative, we can derive two mathematical inequalities in a nonconventional way. In Example 12, $\eta \geq 0$ implies that

$$\mathrm{E}[e^{-aW}]^{1/a} \leq \prod_{i=1}^{n} \mathrm{E}[e^{-a_i W_i}]^{1/a_i}; \tag{189}$$

see (116). With the substitutions

$$Z_i = e^{-a_i W_i}, \qquad r_i = a_i/a,$$

Inequality (189) can be written as

$$\mathrm{E}\left[\prod_{i=1}^{n} Z_i\right] \leq \prod_{i=1}^{n} \mathrm{E}[Z_i^{r_i}]^{1/r_i}. \tag{190}$$

Because the substitutions can be reversed, this inequality is valid for arbitrary random variables $Z_1 > 0, \ldots, Z_n > 0$ and numbers $r_1 > 0, \ldots, r_n > 0$ with $1/r_1 + \ldots + 1/r_n = 1$. In the mathematical literature, Inequality (190) is known as *Hölder's inequality.*

The other inequality is *Minkowski's inequality*. It states that for $p > 1$ and random variables $Z_1 > 0, \ldots, Z_n > 0$ the following inequality holds:

$$\mathrm{E}\left[\left(\sum_{i=1}^{n} Z_i\right)^p\right]^{1/p} \le \sum_{i=1}^{n} \mathrm{E}[Z_i^p]^{1/p}. \qquad (191)$$

The proof starts with $\eta \ge 0$ in (119). Then it suffices to set

$$Z_i = s_i - W_i, \qquad p = c + 1,$$

and to observe that substitutions can be reversed. If $p < 1$, the inequality sign in (191) should be reversed. This follows from Example 14, with the substitution

$$Z_i = W_i, \qquad p = 1 - c.$$

In the limit $p \to 0$, we obtain

$$\exp\left\{\mathrm{E}\left[\ln\left(\sum_{i=1}^{n} Z_i\right)\right]\right\} \ge \sum_{i=1}^{n} \exp\,(\mathrm{E}[\ln Z_i]);$$

this can be seen from (124). Note that (191) also holds for $p = 1$, in which case it is known as the *triangle inequality*.

DISCUSSIONS

HANGSUCK LEE*

Dr. Gerber and Mr. Pafumi have written a very interesting paper. My comments concern Section 7, on the optimal fraction of reinsurance.

In the particular case in which $u(x)$ is an exponential utility function with parameter a and S has a gamma distribution with parameters α and β, the calculation can be also done explicitly. If $S \sim$ gamma(α, β), then $\mathrm{E}(S) = \alpha/\beta$ and $\mathrm{Var}(S) = \alpha/\beta^2$. For $a(1 - \varphi) < \beta$, the expected utility is

$$\frac{1}{a}\,(1 - \mathrm{E}[e^{-aw+a\varphi P+a(1-\varphi)S}])$$

$$= \frac{1}{a}\left(1 - e^{-aw+a\varphi P}\left(\frac{\beta}{\beta - a(1-\varphi)}\right)^{\alpha}\right),$$

which is maximal for

$$1 - \tilde{\varphi} = \left(\beta - \frac{\alpha}{P}\right) \times \frac{1}{a}.$$

To compare this result with the one in normal case, we rewrite it as

$$1 - \tilde{\varphi} = \frac{P - \mathrm{E}(S)}{a\,\mathrm{Var}(S)}\,\frac{\mathrm{E}(S)}{P}.$$

If we assume α, β and P tend to infinity such that $P - \mathrm{E}(S)$ and $\mathrm{Var}(S)$ remain constant, then

$$1 - \tilde{\varphi} \longrightarrow \frac{P - \mathrm{E}(S)}{a\,\mathrm{Var}(S)}.$$

In another particular case in which $u(x)$ is an exponential utility function with parameter a, and S has an inverse Gaussian distribution with parameters α and β (Bowers et al. 1997, Ex. 2.3.5), the calculation can be again done explicitly. If $S \sim$ Inverse Gaussian(α, β), then $\mathrm{E}(S) = \alpha/\beta$ and $\mathrm{Var}(S) = \alpha/\beta^2$. For $a(1 - \varphi) < \beta/2$, the expected utility is

$$\frac{1}{a}\,(1 - \mathrm{E}[e^{-aw+a\varphi P+a(1-\varphi)S}]) = \frac{1}{a}\Bigg(1 - e^{-aw+a\varphi P} \times \exp\left\{\alpha\left[1 - \left(1 - \frac{2a(1-\varphi)}{\beta}\right)^{1/2}\right]\right\}\Bigg).$$

It is maximal for

$$1 - \tilde{\varphi} = \frac{P^2 - (\alpha/\beta)^2}{P^2} \times \frac{\beta}{2a}.$$

To compare this result with that in normal case, we rewrite it as

$$1 - \tilde{\varphi} = \frac{P - \mathrm{E}(S)}{a\,\mathrm{Var}(S)}\,\frac{\left(1 + \frac{\mathrm{E}(S)}{P}\right)}{2}\,\frac{\mathrm{E}(S)}{P}.$$

If we assume α, β and P tend to infinity such that $P - \mathrm{E}(S)$ and $\mathrm{Var}(S)$ are constant, then

$$1 - \tilde{\varphi} \longrightarrow \frac{P - \mathrm{E}(S)}{a\,\mathrm{Var}(S)}.$$

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ALASTAIR G. LONGLEY-COOK*

As Charles Trowbridge (1989, p. 11) points out in *Fundamental Concepts of Actuarial Science,* utility theory forms the philosophical basis of actuarial science, and yet there is barely a mention of the subject in the actuarial literature beyond Chapter 1 of *Actuarial Mathematics* (Bowers et al. 1997). Dr. Gerber and Mr. Pafumi do the profession a great service by publishing this excellent summary of utility functions and their applications.

The authors employ three general forms of utility functions, the exponential and the power of the first and second kind, in their examples. The reasonableness of these functions should be evaluated in any practical application. In particular, the upper and lower bounds on the two kinds of power functions may render them unreasonable assumptions in situations in which results can vary widely from expected.

It is also generally agreed in finance theory that for a utility function to be realistic with regard to economic behavior, its absolute risk aversion with regard to wealth W, defined as $A(W) = -U''(W)/U'(W)$, should be a decreasing function of W; and while there is some debate over the slope of its relative risk aversion, defined as $R(W) = -W[U''(W)/U'(W)]$, empirical evidence suggests that it should be constant over W (Rubinstein 1976). If the variable x in the paper's first example [Formula (1)] is defined as change in wealth W with respect to initial wealth W_0, then the negative exponential utility function satisfies the decreasing-absolute and constant-relative risk aversion criteria with respect to initial wealth W_0. As the authors point out, the absolute risk aversion of the power utility function of the first kind [Formula (10)] *increases* with wealth, a condition that may prove unrealistic.

Note that, despite the differences in utility functions, the general form of a risk adjustment (which reduces the expected value to its certainty equivalent) is dependent on the *variance* of gain, rather than the standard deviation or some other measure or risk [see Formulas (23), (26), and (28)]. This is consistent with mean/variance risk analysis and challenges the use of other risk measures.

What is then surprising is that the classical value for β given in Formula (165) is generated by the quadratic utility function, rather than the exponential. The "expected return versus beta" relationship of the capital-asset-pricing model (CAPM)

$$E(r_i) = r_f + \beta_i[E(r_M) - r_f]$$

can be derived from a variance-proportionate risk premium (Bodie et al. 1996, p. 243)

$$E(r_i) - r_f = k\sigma^2.$$

So it is strange that the exponential utility function assumption, with its variance-proportionate risk premium [see Formula (23)] does not lead to the classical value for β. Further explanation of the utility function assumptions inherent in CAPM would be helpful.

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HEINZ H. MÜLLER*

The authors are to be congratulated for their very elegant article that treats in a very concise and clear style the most important applications of utility theory in insurance and finance. The article not only is an excellent survey but also introduces the synergy potential of risk exchanges as a new concept. In the insurance context this synergy potential measures in a most convincing way the welfare gain resulting from reinsurance.

It may be of some interest to address the question, which types of utility functions lead to a decision-making consistent with empirical observations? The following result due to Arrow (1971) helps to answer this question:

> Assume that a risk-less and a risky asset are available as investment opportunities. Then, under an increase of initial wealth investors with increasing (decreasing) risk aversion decrease (increase) the dollar amount invested in the risky asset.

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Therefore, Arrow postulates a decreasing risk aversion. According to this postulate power utility functions of the first kind and, in particular, the quadratic utility function may not be appropriate for practical purposes.

The results in the survey allow also for some comments on the properties of equilibrium in financial markets. As shown in Section 10, an equilibrium risk exchange $(\tilde{X}_1, \ldots, \tilde{X}_n)$ with a price density Ψ is Pareto optimal, and there exist $k_1, \ldots, k_n$ such that (107) and (109) hold with $\Lambda = \Psi$. This provides the basis for the construction of a fictitious investor representing the market. Up to an additive constant the utility function $\hat{u}$ of this representative investor is given by

$$\hat{u}'(w) = \Psi(w).$$

From (109) we conclude that $\hat{u}$ is strictly concave and for the risk aversion of the representative investor we obtain

$$\hat{r}(w) = -\frac{\hat{u}''(w)}{\hat{u}'(w)} = -\frac{\Psi'(w)}{\Psi(w)} = \frac{1}{\sum_{h=1}^{n} \frac{1}{r_h(\tilde{X}_h)}}.$$

Instead of the risk aversion

$$r(x) = -\frac{u''(x)}{u'(x)},$$

it is convenient in this context to use the risk tolerance, which is defined by

$$\tau(x) = \frac{1}{r(x)}.$$

Hence the risk tolerance of the representative investor is given by

$$\hat{\tau}(w) = \sum_{h=1}^{n} \tau_h(\tilde{X}_h),$$

and the risk-sharing rule (101) can be written as

$$d\tilde{X}_j = \frac{\tau_j(\tilde{X}_j)}{\hat{\tau}(w)}\, dw, \qquad j = 1, \ldots, n$$

or

$$\tilde{X}_j' = \frac{\tau_j(\tilde{X}_j)}{\hat{\tau}(w)}.$$

Taking logarithms and differentiating leads to

$$\frac{\tilde{X}_j''}{\tilde{X}_j'} = \frac{\tau_j'(\tilde{X}_j)}{\tau_j(\tilde{X}_j)}\, \tilde{X}_j' - \frac{\hat{\tau}'(w)}{\hat{\tau}(w)}$$

$$= \frac{1}{\hat{\tau}(w)}\, [\tau_j'(\tilde{X}_j) - \hat{\tau}'(w)].$$

This implies in particular

$$\operatorname{sign}(\tilde{X}_j'') = \operatorname{sign}[\tau_j'(\tilde{X}_j) - \hat{\tau}'(w)],$$

and we conclude

$$\tau_j'(\tilde{X}_j) \underset{(<)}{>} \hat{\tau}'(w) \longrightarrow \tilde{X}_j'' \underset{(<)}{>} 0 \quad (^*).$$

In the context of financial markets, W corresponds to the total market capitalization and $\tilde{X}_j$, $j = 1, \ldots, n$, denotes investor j's payoff as a function of W. According to Arrow's postulate risk tolerances are increasing and (*) can be interpreted as follows:

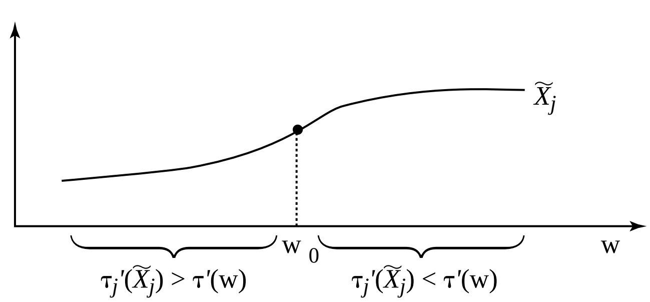
> Investors who are more (less) sensitive to wealth changes than the market choose a convex (concave) payoff function.

As Leland (1980) pointed out in his article on risk-sharing in financial economics, convex payoff functions can be considered as generalized portfolio insurance strategies. A discussion of the shape of payoff functions in market equilibrium can be found, for example, in Chevallier and Mueller (1994).

The relationship (*) may be of some interest for the investment strategy of pension funds. Because of solvency problems, such an investor may be more sensitive to wealth changes than the market for low values of w, that is, $\tau_j'(\tilde{X}_j) > \hat{\tau}(w)$ for $w < w_0$. For high values of w, funding problems disappear and $\tau_j'(\tilde{X}_j) < \hat{\tau}'(w)$ for $w > w_0$ may hold. This leads to a payoff function $\tilde{X}_j$ as depicted in Figure 1.

A payoff function that is convex for low w and concave for high w is, for example, obtained by investing

Figure 1
Payoff Function for a Pension Fund



in the market portfolio, selling calls with high strike prices and buying puts with low strike prices.

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Stanley R. Pliska*

This paper provides a survey of applications of utility theory to selected risk management and insurance problems. I would like to compliment the authors for writing such a clear, interesting, and yet concise treatment of how utility functions can be used for decision-making in the actuarial context. Applications studied range from the fundamental problem faced by a company of setting an insurance premium to esoteric issues such as synergy potentials.

I would also like to complement the authors by extending their exposition in two directions. Both directions have to do with the literature relating utility theory and financial decision-making. The first direction pertains to the utility functions themselves. An important, classical treatment of the use of utility functions for decision-making is by Fishburn (1970). In recent years, however, financial economists such as Bergman (1985), recognizing the limited realism associated with standard utility functions, have developed some generalizations reflecting changing preferences across time. For example, Constantinides (1990) studied a utility function model that reflects habit formation, and Duffie (1992) covered a related notion called recursive utility. It would be interesting to consider how in the actuarial context preferences might change with time.

Another direction the authors could have pursued is the well-known application of utility functions for portfolio management. More than 25 years ago Robert Merton, the recent winner of a Nobel prize in economics, wrote several papers (see his 1990 book) in which for continuous time stochastic process models of asset prices the objective is to maximize expected utility of consumption and/or terminal wealth. He employed a dynamic programming approach, an approach that is elegant yet often impractical due to computational difficulties. More recently, martingale methods have been employed to successfully solve these continuous time optimal portfolio problems [for example, see Pliska (1997) and the book by Boyle et al. (1998), which the authors already reference]. Since maximizing expected utility is the objective preferred by financial economists for managing portfolios of assets and liabilities, the relevance for the actuarial and insurance industries is obvious.

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Elias S.W. Shiu*

This is a masterful paper on utility functions and many of their applications in actuarial science and finance. I am particularly grateful for this paper because I once wrote in *TSA* (Shiu 1993) a defense for the use of utility theory.

We are indebted to the authors for giving a precise explanation of the approximation formula (28). This result can be found in textbooks such as Bowers et al. (1997, Ex. 1.10.a) and Luenberger (1998, p. 256, Ex. 8). The derivation of (28) outlined in these two books combines a second-order approximation with a first-order approximation. Formula (28) probably first appeared in Pratt (1964, Eq. 7). Pratt (1964, p. 125) was rather careful in stating that he assumed the third absolute central moment of G_z to be of smaller order than $\mathrm{Var}[G_z]$; ordinarily, it is of order $(\mathrm{Var}[G_z])^{3/2}$.

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The formulas for the two kinds of power utility functions, as given by (2) and (4), are rather unsymmetrical because there is an s in (2) but not in (4). By modifying (4) as

$$u(x) = \frac{(x - s)^{1-c} - 1}{1 - c}, \qquad x > s, \qquad \text{(D.1)}$$

we can obtain a certain symmetry between the two kinds of utility functions. Then (11) becomes

$$r(x) = \frac{c}{x - s}, \qquad \text{(D.2)}$$

(89) becomes

$$\tilde{X}_j = k_j^{1/c}\, \Lambda^{-1/c} + s_j, \qquad \text{(D.3)}$$

and so on.

My final comment is motivated by the reference to the so-called Merton ratio in Section 7. The Merton ratio was also discussed in a paper in this journal (Boyle and Lin 1997). It means that an investor with a power utility function of the second kind will use a proportional asset investment strategy. The result was derived by Merton (1969) using Bellman's equation. An elegant proof using the insights from the martingale approach to the contingent-claims pricing theory can be found in the review paper by Cox and Huang (1989, p. 283). In the context of discrete-time models, the result was obtained by Mossin (1968); further discussions can be found in survey articles such as Hakansson (1987) and Hakansson and Ziemba (1995) and in various papers reprinted in Ziemba and Vickson (1975). Merton (1969) also showed that an investor with an exponential utility function would invest a constant amount in the risky asset.

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VIRGINIA R. YOUNG*

I congratulate the authors on writing an excellent summary of applications of utility functions in evaluating risks, with special emphasis on insurance economics. Their paper will serve as a valuable reference for actuaries, both practitioners and researchers.

In this discussion, I simply wish to point out that Arrow's theorem on the optimality of stop-loss (or deductible) insurance is more generally true; see the authors' Section 6. Specifically, suppose a decision maker orders risks according to stop-loss ordering; that is, a (non-negative) loss random variable X is considered less risky under stop-loss ordering than a loss Y if

$$\int_0^t S_X(x)dx \le \int_0^t S_Y(x)dx$$

for all $t > 0$, in which S_X is the survival function of X; namely, $S_X(x) = \Pr(X > x)$. Then, for a fixed premium $P = f(\mathrm{E}[h(X)]$, in which f is a function such that $f(x) \ge x$ and $f'(x) \ge 1$, the decision maker will prefer stop-loss insurance with deductible d given implicitly by

$$P = f\left(\int_d^\infty S_X(x)dx\right).$$

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Van Heerwaarden, Kaas, and Goovaerts (1989) prove this result for $f(x) = (1 + \lambda)x$, while constraining the indemnity benefit to have a derivative between zero and one. The result is true when one only constrains the indemnity benefit to lie between zero and the loss amount, as in the authors' Inequality (54); see Gollier and Schlesinger (1996).

The optimality of stop-loss insurance is intuitive by noting that the authors' proof of Inequality (56) is independent of the (increasing, concave) utility function and by recalling that the common (partial) ordering of random variables by risk-averse decision makers is stop-loss ordering (Wang and Young 1998). I encourage the authors and interested researchers to explore whether or not one can generalize other results from expected utility theory.

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AUTHORS' REPLY

HANS U. GERBER AND GÉRARD PAFUMI

Mr. Lee shows how the optimal degree of proportional reinsurance can be determined explicitly if S has a gamma or an inverse Gaussian distribution. He also shows that Formula (60) is obtained in both cases as a limiting result. This raises the question, What results can be obtained under the more general assumption that S has an infinitely divisible distribution? Let

$$\theta(z) = z + \frac{1}{2}z^2 + \frac{\gamma}{6}z^3 + \ldots$$

denote the cumulant generating function of a random variable with infinitely divisible distribution, having mean 1, variance 1, and third central moment γ. Now suppose that S has a distribution such that its moment generating function is

$$M_S(z) = E[e^{zS}] = e^{\alpha\theta(z/\beta)} \tag{R.1}$$

for some $\alpha > 0$ (the shape parameter) and $\beta > 0$ (the scale parameter). Then

$$E[S] = \frac{\alpha}{\beta},$$

$$\text{Var}[S] = \frac{\alpha}{\beta^2},$$

$$E\left[\left(S - \frac{\alpha}{\beta}\right)^3\right] = \gamma\frac{\alpha}{\beta^3}.$$

This construction is illustrated by the following table.

Distribution	$\theta(z)$	γ
Normal	$z + \frac{1}{2}z^2$	0
Gamma	$-\ln(1 - z)$	2
Inverse Gaussian	$1 - \sqrt{1 - 2z}$	3

The expected utility (59) is

$$= \frac{1}{a}(1 - \exp(-aw + a\varphi P)M_S(a(1 - \varphi))).$$

Using (R.1), we see that we must minimize the expression

$$a\varphi P + \alpha\theta\left(\frac{a(1 - \varphi)}{\beta}\right).$$

Setting the derivative equal to 0, we gather that $\tilde{\varphi}$ is obtained from the condition

$$P - \frac{\alpha}{\beta}\theta'\left(\frac{a(1 - \tilde{\varphi})}{\beta}\right) = 0. \tag{R.2}$$

In general, there is no explicit formula for $1 - \tilde{\varphi}$. However, it is possible to obtain an asymptotic formula. Let $P \to \infty$, $\alpha \to \infty$, $\beta \to \infty$, such that

$$P - \frac{\alpha}{\beta} = P - E[S] = \text{constant},$$

$$\frac{\alpha}{\beta^2} = \text{Var}[S] = \text{constant}.$$

Substituting $\theta'(z) = 1 + z + (\gamma/2)z^2 + \ldots$ in (R.2), we get

$$P - E[S] - a(1 - \tilde{\varphi})\text{Var}[S]$$
$$- \frac{\gamma a^2}{2\beta}(1 - \tilde{\varphi})^2\,\text{Var}[S] + \ldots = 0.$$

Finally, we develop $1 - \tilde{\varphi}$ in powers of $1/\beta$ and obtain the formula

$$1 - \tilde{\varphi} = \frac{P - E[S]}{a\,\text{Var}[S]}\left\{1 - \frac{\gamma}{2\beta}\frac{P - E[S]}{\text{Var}[S]} + \ldots\right\}.$$

Again, Formula (60) results in the limit.

We appreciate the comments of Mr. Longley-Cook. He finds it strange that the assumption of exponential utility functions does not lead to the classical expression for β. However, if Formula (162) is expanded in powers of a, the classical Formula (165) can be obtained as a first-order approximation. In this context we note that Formulas (23), (26), and (28) are also first-order approximations. For a deeper discussion of the CAPM formula, refer to Section 8.2.4 of Panjer et al. (1998) and the references quoted therein.

Dr. Müller raises some very interesting points. The risk tolerance function (the reciprocal of the risk aversion function) leads to simplification of some formulas and is a useful and appealing tool by its own.

Dr. Pliska and Dr. Shiu point out an important application of utility theory: the construction of an optimal portfolio, that is, a portfolio that maximizes the expected utility of an investor. This problem can indeed be discussed in the framework of Sections 10 and 11 of the paper. Consider an investor with wealth w at time 0 and utility function $u(x)$ to assess the terminal wealth at time T. Let $\delta > 0$ denote the riskless force of interest. In the market random payments can be bought. Their price is given by a price density Ψ. Thus the price (due at time 0) for a payment of Y (due at time T) is $e^{-\delta T} E[\Psi Y]$. If the investor buys Y, the terminal wealth will be

$$W_T = we^{\delta T} + Y - E[\Psi Y]. \qquad \text{(R.3)}$$

The problem is to choose Y that maximizes $E[u(W_T)]$. In analogy to (141), the solution is characterized by the condition

$$u'(W_T) = \Psi E[u'(W_T)] \qquad \text{(R.4)}$$

with W_T given by (R.3). For the utility functions of Examples 1 to 3, explicit expressions for the optimal terminal wealth are obtained. For an exponential utility function, $u'(x) = e^{-ax}$, the optimal terminal wealth is

$$W_T = we^{\delta T} - \frac{1}{a}\ln\Psi + \frac{1}{a}E[\Psi\ln\Psi]. \qquad \text{(R.5)}$$

For a power utility function of the first kind, $u'(x) = (1 - x/s)^c$, $x < s$, we obtain

$$W_T = s - \frac{s - we^{\delta T}}{E[\Psi^{1+1/c}]}\Psi^{1/c}, \qquad \text{(R.6)}$$

and for a power utility function of the second kind, $u'(x) = x^{-c}$, the result is that

$$W_T = \frac{we^{\delta T}}{E[\Psi^{1-1/c}]}\Psi^{-1/c}. \qquad \text{(R.7)}$$

We note that W_T is the solution of a *static* optimization problem. If we make appropriate additional assumptions about the market, we can determine the optimal investment strategy, that is, the *dynamic* strategy that replicates the optimal terminal wealth W_T. As in Section 10.6 of Panjer et al. (1998), assume that two securities are traded continuously, the riskless investment (which grows at a constant rate δ), and a non-dividend-paying stock, with price $S(t)$ at time t, $0 \le t \le T$. We make the classical assumption that $\{S(t)\}$ is a geometric Brownian motion, that is,

$$S(t) = S(0)e^{X(t)}$$

where $\{X(t)\}$ is a Wiener process with parameters μ and σ^2 and parameters $\mu^* = \delta - (1/2)\sigma^2$ and σ^2 in the risk-neutral measure. Then

$$\Psi = e^{\alpha T}\left(\frac{S(T)}{S(0)}\right)^{h^*}, \qquad \text{with } h^* = \frac{\mu^* - \mu}{\sigma^2}, \qquad \text{(R.8)}$$

where α is such that $E[\Psi] = 1$. Note that h^* is defined as in Gerber and Shiu (1994) and Gerber and Shiu (1996), that is, as the value of the Esscher parameter h, for which the discounted stock price process is a martingale under the transformed measure. As a preparation, we recall a result concerning the self-financing portfolio that replicates the payoff of a European-type contingent claim. Consider a European contingent claim with terminal date T and payoff function $\Pi(z)$; that is, at time T the payoff $\Pi(S(T))$ is due. Let $V(z, t)$ denote its price at time t, and $\eta(z, t)$ the amount invested in stocks in the replicating portfolio at time t, if $S(t) = z$. It is well known that

$$\eta(z, t) = z\frac{\partial V(z, t)}{\partial z}; \qquad \text{(R.9)}$$

see Formula (10.6.6) in Panjer et al. (1998), page 95 of Baxter and Rennie (1996), or Section 9.3 of Dothan (1990).

Let us revisit the three examples. From (R.5) and (R.8) we obtain

$$W_T = we^{\delta T} + \frac{1}{a}E[\Psi\ln\Psi] - \frac{\alpha T}{a} - \frac{h^*}{a}\ln S(T) + \frac{h^*}{a}\ln S(0). \qquad \text{(R.10)}$$

Consider a European contingent claim with terminal date T and payoff function

$$\prod(z) = we^{\delta T} + \frac{1}{a}E[\Psi\ln\Psi] - \frac{\alpha T}{a} - \frac{h^*}{a}\ln z.$$

Its payoff differs from W_T only by the constant

$-(h^*/a)\ln S(0)$. Since w is the initial price of W_T, it follows that

$$V(z, 0) = w - \frac{h^*}{a} e^{-\delta T} \ln z.$$

Hence, by (R.9), the initial amount invested in stocks must be

$$\eta(z, 0) = -\frac{h^*}{a} e^{-\delta T}.$$

Similarly, at time t, the amount invested in stocks in the replicating portfolio is

$$-\frac{h^*}{a} e^{-\delta(T-t)} = \frac{\mu - \mu^*}{a\sigma^2} e^{-\delta(T-t)}, \; 0 \le t \le T. \quad \text{(R.11)}$$

Note that its discounted value is constant.

For a power utility function of the first kind, we gather from (R.6) and (R.8) that the optimal terminal wealth is

$$W_T = s - \frac{s - we^{\delta T}}{E[\Psi^{1+1/c}]} e^{\alpha T/c} \left(\frac{S(T)}{S(0)}\right)^{h^*/c}.$$

Now consider a European contingent claim with terminal date T and payoff function

$$\prod(z) = \frac{s - we^{\delta T}}{E[\Psi^{1+1/c}]} \exp(\alpha T/c) z^{h^*/c}.$$

Note that

$$\prod(S(T)) = (s - W_T) S(0)^{h^*/c}. \quad \text{(R.12)}$$

It follows that the initial price of the contingent claim is

$$V(z, 0) = (se^{-\delta T} - w) z^{h^*/c},$$

with $S(0) = z$. Then according to (R.9) we have

$$\eta(z, 0) = \frac{h^*}{c} (se^{-\delta T} - w) z^{h^*/c}. \quad \text{(R.13)}$$

To determine the replicating portfolio for W_T, we rewrite (R.12) as

$$W_T = s - \prod(S(T)) S(0)^{-h^*/c}.$$

Hence the amount invested in stocks at time 0 must be

$$-\eta(S(0), 0)\, S(0)^{-h^*/c}$$

which, by (R.13), simplifies to

$$-\frac{h^*}{c} (se^{-\delta T} - w).$$

Similarly, at time t the amount invested in stocks in the replicating portfolio is

$$-\frac{h^*}{c} (se^{-\delta(T-t)} - W_t) = \frac{\mu - \mu^*}{c\sigma^2} (se^{-\delta(T-t)} - W_t), \quad \text{(R.14)}$$

where W_t is the wealth at time t, a constant fraction of what is missing for total satisfaction.

For a power utility function of the second kind, the optimal terminal wealth is

$$W_T = \frac{we^{\delta T}}{E[\Psi^{1-1/c}]} e^{-\alpha T/c} \left(\frac{S(T)}{S(0)}\right)^{-h^*/c}.$$

This time consider a European contingent claim with terminal date T and payoff function

$$\prod(z) = \frac{we^{\delta T}}{E[\Psi^{1-1/c}]} \exp(-\alpha T/c) z^{-h^*/c}.$$

Its price at time 0 is

$$V(z, 0) = z^{-h^*/c} w.$$

Hence, by (R.9),

$$\eta(z, 0) = -\frac{h^*}{c} V(z, 0).$$

For the replicating portfolio of W_T, the amount invested in stocks is the same constant fraction of total wealth, that is,

$$-\frac{h^*}{c} w$$

at time 0, and

$$-\frac{h^*}{c} W_t = \frac{\mu - \mu^*}{c\sigma^2} W_t \quad \text{(R.15)}$$

at time t.

At first sight, expressions (R.11), (R.14), and (R.15) are quite different. However, they can be written in a common form: in all three cases the optimal trading strategy is to invest the amount

$$\frac{\mu - \mu^*}{\sigma^2 r(e^{\delta(T-t)} W_t)} e^{-\delta(T-t)} \quad \text{(R.16)}$$

at time t ($0 \le t < T$) in stocks, where r is the risk aversion function. For a verification, simply use (9), (10), and (11) of the paper.

These results can be generalized to the case where $n \ge 2$ different types of stocks are traded. Let $S_k(t)$ denote the price of stock k. We assume that

$\{S_1(t), \ldots, S_n(t)\}$ is an n-dimensional geometric Brownian motion with drift parameters $\mu_1, \ldots, \mu_n$ ($\mu_1^*, \ldots, \mu_n^*$ in the risk-neutral measure) and covariances σ_{ik}. It is assumed that the covariance matrix has an inverse (the precision matrix); its elements are denoted by the symbol τ_{ik}. Then we find the following generalization of (R.8):

$$\Psi = e^{\alpha T} \prod_{k=1}^{n} \left(\frac{S_k(T)}{S_k(0)}\right)^{h_k^*}, \qquad \text{with } h_k^* = \sum_{i=1}^{n} (\mu_i^* - \mu_i)\tau_{ik}$$

defined as in Section 7 of Gerber and Shiu (1994), and again α such that $E[\Psi] = 1$. According to (R.5), (R.6), (R.7), the optimal terminal wealth is

$$W_T = we^{\delta T} + \frac{1}{a} E[\Psi \ln \Psi] - \frac{\alpha T}{a} - \frac{1}{a} \sum_{k=1}^{n} h_k^* \ln S_k(T) + \frac{1}{a} \sum_{k=1}^{n} h_k^* \ln S_k(0)$$

for an exponential utility function,

$$W_T = s - \frac{s - we^{\delta T}}{E[\Psi^{1+1/c}]} e^{\alpha T/c} \prod_{k=1}^{n} \left(\frac{S_k(T)}{S_k(0)}\right)^{h_k^*/c}$$

for a power utility function of the first kind and

$$W_T = \frac{we^{\delta T}}{E[\Psi^{1-1/c}]} e^{-\alpha T/c} \prod_{k=1}^{n} \left(\frac{S_k(T)}{S_k(0)}\right)^{-h_k^*/c}$$

for a power utility function of the second kind. In each case we can relate the optimal terminal wealth to the payoff of an appropriately chosen European contingent claim. Such a payoff can be replicated by a dynamic portfolio, whereby the amount η_k is invested in stocks of type k at time t. Let $V(z_1, \ldots, z_n, t)$ denote the price of the portfolio at time t (if $S_k(t) = z_k$, $k = 1, \ldots, n$). Then

$$\eta_k(z_1, \ldots, z_n, t) = z_k \frac{\partial V(z_1, \ldots, z_n, t)}{\partial z_k}$$

for $k = 1, \ldots, n$. See, for example, Formula (8.35) in Gerber and Shiu (1996). The portfolio that replicates W_T is the optimal investment strategy. For an exponential utility function, we find that the amount

$$-\frac{h_k^*}{a} e^{-\delta(T-t)} \tag{R.17}$$

must be invested in stocks of type k at time t. For a power utility function of the first kind, the corresponding amount is

$$-\frac{h_k^*}{c} (se^{-\delta(T-t)} - W_t), \tag{R.18}$$

and for a power utility function of the second kind, the amount invested in stocks of type k at time t is

$$-\frac{h_k^*}{c} W_t. \tag{R.19}$$

Again, (R.17), (R.18), and (R.19) can be written in a common form. Now the optimal trading strategy consists of investing the amount

$$-\frac{h_k^*}{r(e^{\delta(T-t)} W_t)} e^{-\delta(T-t)} = \frac{\sum_{i=1}^{n} (\mu_i - \mu_i^*)\tau_{ik}}{r(e^{\delta(T-t)} W_t)} e^{-\delta(T-t)} \tag{R.20}$$

of stock of type k at time t. It follows that the total amount invested in stocks at time t is

$$\frac{\sum_{k=1}^{n} \sum_{i=1}^{n} (\mu_i - \mu_i^*)\tau_{ik}}{r(e^{\delta(T-t)} W_t)} e^{-\delta(T-t)}. \tag{R.21}$$

Hence at any time the amount invested in stock of type k must be the constant fraction

$$\frac{\sum_{i=1}^{n} (\mu_i - \mu_i^*)\tau_{ik}}{\sum_{k=1}^{n} \sum_{i=1}^{n} (\mu_i - \mu_i^*)\tau_{ik}} \tag{R.22}$$

of the total amount invested in stocks. Note that this fraction does not depend on the utility function.

If we divide expressions (R.16), (R.20), or (R.21) by W_t, we obtain the Merton ratios. For more results and further background, refer to Chapter 8 of Duffie (1992) and the annotated references.

Needless to say, we share Dr. Shiu's enthusiasm for utility functions. We were pleased to see that utility-related papers by Longley-Cook (1998) and Frees (1998) have been published by the *NAAJ*. Dr. Shiu proposes a more symmetric treatment of power utility functions. The utility function in his Formula (D.1) is standardized at the point $\xi = 1 + s$. If $s < 0$, it may be natural to standardize it at $\xi = 0$, which yields the formula

$$u(x) = \frac{(x - s)^{1-c} - (-s)^{1-c}}{(1 - c)(-s)^{-c}}, \qquad x > s.$$

Then, in the limit $s \to -\infty$, we obtain $u(x) = x$.

Dr. Young points out a generalization of Arrow's result concerning optimality of a stop-loss coverage. The assumption that the premium for $h(X)$ should depend only on $E[h(X)]$ is surprising but crucial for the conclusion. A result for more realistic premium calculation principles is given in Theorem 9 of Deprez and Gerber (1985). For example, if the company has an exponential utility function with parameter b and if the reinsurance premiums are calculated according to the exponential premium principle with parameter $a > 0$, the exponential utility is maximized for $h(S) = \varphi S$ (proportional coverage) with $\varphi = b/(a + b)$. More general results can be found in Young (1998).

We would like to add a pedagogical comment. It is possible to proceed in Examples 20 and 21 the same way as in Example 19, that is, by first determining the net demand of company i. In Example 20, it is

$$-W_i + s_i - \frac{s_i - E[\Psi W_i]}{E[\Psi^{1+1/c}]} \Psi^{1/c},$$

and in Example 21, it is

$$-W_i + \frac{E[\Psi W_i]}{E[\Psi^{1-1/c}]} \Psi^{-1/c}.$$

For an equilibrium, the sum over i must vanish, which, together with $E[\Psi] = 1$ yields (155) and (158) of the paper.

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