# Applications of Option-Pricing Theory: Twenty-Five Years Later

Robert C. Merton

# Applications of Option-Pricing Theory: Twenty-Five Years Later[†]

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The news from Stockholm that the prize in economic sciences had been given for option-pricing theory provided unique and signal recognition to the rapidly advancing, but still relatively new, discipline within economics which relates mathematical finance theory and finance practice.[1] The special sphere of finance within economics is the study of allocation and deployment of economic resources, both spatially and across time, in an uncertain environment. To capture the influence and interaction of time and uncertainty effectively requires sophisticated mathematical and computational tools. Indeed, mathematical models of modern finance contain some truly elegant applications of probability and optimization theory. These applications challenge the most powerful computational technologies. But, of course, all that is elegant and challenging in science need not also be practical; and surely, not all that is practical in science is elegant and challenging. Here we have both. In the time since publication of our early work on the option-pricing model, the mathematically complex models of finance theory have had a direct and wide-ranging influence on finance practice. This conjoining of intrinsic intellectual interest with extrinsic application is central to research in modern finance.

It was not always thus. The origins of much of the mathematics in modern finance can be traced to Louis Bachelier's 1900 dissertation on the theory of speculation, framed as an option-pricing problem. This work marks the twin births of both the continuous-time mathematics of stochastic processes and the continuous-time economics of derivative-security pricing. Kiyoshi Itô (1987) was greatly influenced by Bachelier's work in his development in the 1940's and early 1950's of the stochastic calculus, later to become an essential mathematical tool in finance. Paul A. Samuelson's theory of rational warrant pricing, published in 1965, was also motivated by the same piece. However, Bachelier's important work was largely lost to financial economists for more than a half century. During most of that period, mathematically complex models with a strong influence on practice were not at all the hallmarks of finance theory. Before the pioneering work of Markowitz, Modigliani, Miller, Sharpe, Lintner, Fama, and Samuelson in the late 1950's and 1960's, finance theory was little more than a collection of anecdotes, rules of thumb, and shuffling of accounting data. It was not until the end of the 1960's and early 1970's that models of finance in academe become considerably more sophisticated, involving both the intertemporal and uncertainty dimensions of valuation and optimal decision-making. The new models of dynamic portfolio theory, intertemporal capital asset pricing, and derivative-security pricing employed stochastic differential and integral equations, stochastic dynamic programming, and partial differential equations. These mathematical tools were a quantum level more complex than had been used in finance before and they are still the core tools employed today.

The most influential development in terms of impact on finance practice was the Black-Scholes model for option pricing. Yet

[1] This section draws on Merton (1994, 1995, 1997b).

paradoxically, the mathematical model was developed entirely in theory, with essentially no reference to empirical option-pricing data as motivation for its formulation. Publication of the model brought the field to almost immediate closure on the fundamentals of option-pricing theory. At the same time, it provided a launching pad for refinements of the theory, extensions to derivative-security pricing in general, and a wide range of other applications, some completely outside the realm of finance. The Chicago Board Options Exchange (CBOE), the first public options exchange, began trading in April 1973, and by 1975, traders on the CBOE were using the model to both price and hedge their option positions. It was so widely used that, in those pre-personal-computer days, Texas Instruments sold a handheld calculator specially programmed to produce Black-Scholes option prices and hedge ratios. That rapid adoption was all the more impressive, as the mathematics used in the model were not part of the standard mathematical training of either academic economists or practitioner traders.

Academic finance research of the 1960's, including capital asset pricing, performance, and risk measurement, and the creation of the first large-scale databases for security prices essential for serious empirical work, have certainly influenced subsequent finance practice. Still the speed of adoption and the intensity of that influence was not comparable to the influence of the option model. There are surely several possible explanations for the different rates of adoption in the 1960's and the 1970's. My hypothesis is that manifest "need" determined that difference. In the 1960's, especially in the United States, financial markets exhibited unusually low volatility: the stock market rose steadily, interest rates were relatively stable, and exchange rates were fixed. Such a market environment provided investors and financial service firms with little incentive to adopt new financial technology, especially technology designed to help manage risk. However, the 1970's experienced several events that caused both structural changes and large increases in volatility. Among the more important events were: the shift from fixed to floating exchange rates with the fall of Bretton Woods and the devaluation of the dollar; the

world oil-price shock with the creation of OPEC; double-digit inflation and interest rates in the United States; and the extraordinary real-return decline in the U.S. stock market from a peak of around 1050 on the Dow Jones Industrial Average in the beginning of 1973 to about 580 at the end of 1974. As a result, the increased demand for managing risks in a volatile and structurally different economic environment contributed to the major success of the derivative-security exchanges created in the 1970's to trade listed options on stocks, futures on major currencies, and futures on fixed-income instruments. This success in turn increased the speed of adoption for quantitative financial models to help value options and assess risk exposures.

The influence of option-pricing theory on finance practice has not been limited to financial options traded in markets or even to derivative securities generally. As we shall see, the underlying conceptual framework originally used to derive the option-pricing formula can be used to price and evaluate the risk in a wide array of applications, both financial and nonfinancial. Option-pricing technology has played a fundamental role in supporting the creation of new financial products and markets around the globe. In the present and in the impending future, that role will continue expanding to support the design of entirely new financial institutions, decision-making by senior management, and the formulation of public policy on the financial system. To underscore that point, I begin with a few remarks about financial innovation of the past, this adumbration to be followed in later sections with a detailed listing of applications of the options technology that include some observations on the directions of future changes in financial services.

New financial product and market designs, improved computer and telecommunications technology, and advances in the theory of finance during the past quarter century have led to dramatic and rapid changes in the structure of global financial markets and institutions. The scientific breakthroughs in financial modeling in this period both shaped and were shaped by the extraordinary flow of financial innovation which coincided with those changes. Thus, the publication of the option-

pricing model in 1973 surely helped the development and growth of the listed options and over-the-counter (OTC) derivatives markets. But, the extraordinary growth and success of those markets just as surely stimulated further development and research focus on the derivative-security pricing models. To see this in perspective, consider some of the innovative changes in market structure and scale of the global financial system since 1973. There occurred the aforementioned fall of Bretton Woods leading to floating exchange rates for currencies; the development of the national mortgage market in the United States which in turn restructured that entire industry; passage of the Employee Retirement Income Security Act (ERISA) in 1974 with the subsequent development of the U.S. pension-fund industry; the first money-market fund with check writing that also took place in 1974; and the explosive growth in mutual fund assets from $48 billion 25 years ago to $4.3 trillion today (a 90-fold increase), with one institution, Fidelity Investments, accounting for some $500 billion by itself. In this same period, average daily trading volume on the New York Stock Exchange grew from 12 million shares to more than 300 million. Even more dramatic were the changes in Europe and in Asia. The cumulative impact has significantly affected all of us — as users, producers, or overseers of the financial system.

Nowhere has this been more the case than in the development, refinement, and broad-based implementation of contracting technology. Derivative securities such as futures, options, swaps, and other contractual agreements—the underlying substantive instruments for which our model was developed—provide a prime example. Innovations in financial-contracting technology have improved efficiency by expanding opportunities for risk sharing, lowering transactions costs, and reducing information and agency costs. The numbers reported for the global use of derivative securities are staggering (the figure of $70 trillion appeared more than once in the news stories surrounding the award of the Nobel Prize and there are a number of world banking institutions with reported multi-trillion-dollar, off-balance-sheet derivative positions). However, since these are notional amounts (and often involve double-counting), they are meaningless for assessing either the importance or the risk exposure to derivative securities.[2] Nevertheless, it is enough to say here that, properly measured, derivatives are ubiquitous throughout the world financial system and that they are used widely by nonfinancial firms and sovereigns as well as by institutions in virtually every part of their financing and risk-managing activities. Some observers see the extraordinary growth in the use of derivatives as fad-like, but a more likely explanation is the vast saving in transactions costs derived from their use. The cost of implementing financial strategies for institutions using derivatives can be one-tenth to one-twentieth of the cost of executing them in the underlying cash-market securities.[3] The significance of reducing spread costs in financing can be quite dramatic for corporations and for sovereigns: for instance, not long ago, a 1-percent (i.e., 100-basis-point) reduction in debt-spread cost on Italian government debt would have reduced the deficit by an amount equal to 1.25 percent of the gross domestic product of Italy.

Further improved technology, together with growing breadth and experience in the applications of derivatives, should continue to reduce transactions costs as both users and producers of derivatives move along the learning curve. Like retail depositors with automatic-teller machines in banks, initial resistance by institutional clients to contractual agreements can be high, but once customers use them they tend not to return to the traditional alternatives for implementing financial strategies.

A central process in the past two decades has been the remarkable rate of globalization of the financial system. Even today, inspection

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[2] Notional amounts typically represent either the total value of the underlying asset on which payments on the derivative is determined (e.g., interest rate swap contracts) or the exercise price on an option. The value of the derivative contract itself is often a small fraction of its notional amount.

[3] See André F. Perold (1992) for a case study illustrating the savings in transactions costs, taxes, and custodial fees from using derivatives instead of the cash market. Myron S. Scholes (1976) provides an early analysis of the effect of taxes on option prices.

of the diverse financial systems of individual nation-states would lead one to question how effective integration across geopolitical borders could have realistically taken place since those systems are rarely compatible in institutional forms, regulations, laws, tax structures, and business practices. Still, significant integration did take place. This was made possible in large part by derivative securities functioning as "adapters." In general, the flexibility created by the widespread use of contractual agreements, other derivatives, and specialized institutional designs provides an offset to dysfunctional institutional rigidities.[4] More specifically, derivative-security contracting technologies provide efficient means for creating cross-border interfaces among otherwise incompatible domestic systems, without requiring widespread or radical changes within each system. For that reason, implementation of derivative-security technology and markets within smaller and emerging-market countries may help form important gateways of access to world capital markets and global risk sharing. Such developments and changes are not limited only to the emerging-market countries with their new financial systems. Derivatives and other contracting technologies are likely to play a significant role in the financial engineering of the major transitions required for European Monetary Union and for the major restructuring of financial institutions in Japan.

With this introduction as background, I turn now to the key conceptual and mathematical framework underlying the option-pricing model and its subsequent applications.

## I. General Derivation of Derivative-Security Pricing

I understand that it is customary in these lectures for the Nobel Laureates to review the background and the process leading up to their discoveries. Happily, there is no need to do so here since that has been done elsewhere in Fischer Black (1989), Peter L. Bernstein (1992 Ch. 11), Merton and Scholes (1995), and Scholes (1998). Instead, I briefly summarize. My principal contribution to the Black-Scholes option-pricing theory was to show that the dynamic trading strategy prescribed by Black and Scholes to offset the risk exposure of an option would provide a perfect hedge in the limit of continuous trading. That is, if one could trade continuously without cost, then following their dynamic trading strategy using the underlying traded asset and the riskless asset would exactly replicate the payoffs on the option. Thus, in a continuous-trading financial environment, the option price must satisfy the Black-Scholes formula or else there would be an opportunity for arbitrage profits. To demonstrate this limit-case result, I applied the tools developed in my earlier work (1969, 1971) on the continuous-time theory of portfolio selection. My 1973a paper also extended the applicability of the Black-Scholes model to allow for stochastic interest rates on the riskless asset, dividend payments on the underlying asset, a changing exercise price, American-type early-exercise of the option, and other "exotic" features such as the "down-and-out" provision on the option. I am also responsible for naming the model, "the Black-Scholes Option-Pricing Model."[5]

The derivations of the pricing formula in both of our 1973 papers make the following assumptions:

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[4] Scholes and Mark A. Wolfson (1992) develop the principles of security and institutional design along these lines. See also Perold (1992) and Merton (1993, 1995). Inspection of the weekly *International Financing Review* will find the widespread and varied applications of financial engineering, derivatives, special-purpose vehicles, and securities for private-sector and sovereign financing in every part of the world.

[5] My 1970 working paper was the first to use the "Black-Scholes" label for their model (cf., Merton 1992 p. 379). This same paper was given at the July 1970 Wells Fargo Capital Market Conference, since made "famous" (or notorious) by Bernstein (1992 p. 223) as the one at which I ". . . inconveniently overslept . . ." the morning session and missed the Black and Scholes presentation. The second instance naming their model was in the 1971 working paper version of Merton (1973a). Samuelson (1972) is the first published usage: both in the main text and in my Appendix to that paper which derives the model and refers to it as the "Black-Scholes formula." The formula is cited in Roger J. Leonard (1971) and Carliss Baldwin (1972), the earliest theses to apply the model. Somewhat ironically, all these references to the "Black-Scholes model" appear before the actual publication of either Black and Scholes (1972) or (1973).

ASSUMPTION 1: *"Frictionless" and "continuous" markets*—*There are no transactions costs or differential taxes. Markets are open all the time and trading takes place continuously. Borrowing and short-selling are allowed without restriction. The borrowing and lending rates are equal.*

ASSUMPTION 2: *Underlying asset-price dynamics*—*Let $V = V(t)$ denote the price at time $t$ of a limited-liability asset, such as a share of stock. The posited dynamics for the instantaneous returns can be described by an Itô-type stochastic differential equation with continuous sample paths given by*

$$dV = [\alpha V - D_1(V, t)]\,dt + \sigma V dZ,$$

*where: $\alpha \equiv$ instantaneous expected rate of return on the security; $\sigma^2 \equiv$ instantaneous variance rate, which is assumed to depend, at most, on $V(t)$ and $t$ (i.e., $\sigma^2 = \sigma^2(V, t)$); $dZ$ is a Wiener process; and $D_1 \equiv$ dividend payment flow rate. With limited liability, to avoid arbitrage, $V(t) = 0$ for all $t \geq t^*$ if $V(t^*) = 0$. Hence $D_1$ must satisfy $D_1(0, t) = 0$. Other than a technical requirement of bounded variation, $\alpha$ can follow a quite general stochastic process, dependent on $V$, other security prices, or state variables. In particular, the assumed dynamics permit a mean-reverting process for the underlying asset's returns.*

ASSUMPTION 3: *Default-free bond-price dynamics*—*Bond returns are assumed to be described by Itô stochastic processes with continuous sample paths. In the original Black and Scholes formulation and for exposition convenience here, it is assumed that the riskless instantaneous interest rate, $r(t) = r$, is a constant over time.*

ASSUMPTION 4: *Investor preferences and expectations*—*Investor preferences are assumed to prefer more to less. All investors are assumed to agree on the function $\sigma^2$ and on the Itô process characterization for the return dynamics. It is not assumed that they agree on the expected rate of return, $\alpha$.*

ASSUMPTION 5: *Functional dependence of the option-pricing formula*—*The option price*

*is assumed to be a twice-continuously differentiable function of the asset price, $V$, default-free bond prices, and time.*

In the particular case of a nondividend-paying asset ($D_1 = 0$) and a constant variance rate, $\sigma^2$, these assumptions lead to the Black-Scholes option-pricing formula for a European-type call option with exercise price $L$ and expiration date $T$, written as

$$(1) \quad C(V,t) = VN(d) - L\exp(-r[T-t])$$
$$\times N(d - \sigma\sqrt{T-t}),$$

where $d = (\ln[V/L] + [r + \sigma^2/2][T - t])/\sigma\sqrt{T-t}$ and $N(\ )$ *is the cumulative density function for the standard normal distribution.*

Subsequent research in the field proceeded along three dimensions: applications of the technology to other than financial options (which is discussed in the next section); empirical testing of the pricing formula, which began with a study using over-the-counter data from a dealer's book obtained by Black and Scholes (1972); and attempts to weaken the assumptions used in the derivation, and thereby to strengthen the foundation of the applications developed from this research. The balance of this section addresses issues of the latter dimension.

Early concerns raised about the model's theoretical foundation came from John B. Long (1974) and Clifford W. Smith, Jr. (1976), who questioned Assumption 5: namely, how does one know that the option prices do not depend on other variables than the ones assumed (for instance, the price of beer), and why should the pricing function be twice-continuously differentiable? These concerns were resolved in an alternative derivation in Merton (1977b) which shows that Assumption 5 is a derived consequence, not an assumption, of the analysis.[6]

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[6] As another instance of early questioning of the core model, a paper I refereed argued that Black-Scholes must be fundamentally flawed because a different valuation formula is derived from the replication argument if the R. L. Stratonovich (1968) stochastic calculus is used for mod-

A broader, and still open, research issue is the robustness of the pricing formula in the absence of a dynamic portfolio strategy that exactly replicates the payoffs to the option security. Obviously, the conclusion on that issue depends on why perfect replication is not feasible as well as on the magnitude of the imperfection. Continuous trading is, of course, only an idealized prospect, not literally obtainable; therefore, with discrete trading intervals, replication is at best only approximate. Subsequent simulation work has shown that within the actual trading intervals available and the volatility levels of speculative prices, the error in replication is manageable, provided, however, that the other assumptions about the underlying process obtain. John C. Cox and Stephen A. Ross (1976) and Merton (1976a, b) relax the continuous sample-path assumption and analyze option pricing using a mixture of jump and diffusion processes to capture the prospect of nonlocal movements in the underlying asset's return process.[7] Without a continuous sample path, replication is not possible and that rules out a strict no-arbitrage derivation. Instead, the derivation of the option-pricing model is completed by using equilibrium asset pricing models such as the Intertemporal CAPM (Merton, 1973b) and the Arbitrage Pricing Theory (Ross, 1976a).[8] This approach relates back to the original way in which Black and Scholes derived their model using the classic Sharpe-Lintner

CAPM.[9] There has developed a considerable literature on the case of imperfect replication [cf., Breeden (1984), Hans Föllmer and Dieter Sondermann (1986), Steven Figlewski (1989), Dimitris Bertsimas et al. (1997), Mark H. A. Davis (1997), and Marc Romano and Nizar Touzi (1997)].

On this occasion, I reexamine the imperfect-replication problem for a derivative security linked to an underlying asset that is not continuously available for trading in an environment in which some assets are tradable at any time. As is discussed in the section to follow, non-tradability is the circumstance for several important classes of applications that have evolved over the last quarter century, which include among others, the pricing of financial guarantees such as deposit and pension insurance and the valuation of nonfinancial or "real" options. Since the Black-Scholes model was derived by assuming that the underlying asset is continuously traded, questions have been raised about whether the pricing formula can be properly applied in those applications. The derivation follows along the lines presented in Merton (1977b, 1997b) for the perfect-replication case.

A *derivative security* has contractually determined payouts that can be described by functions of observable asset prices and time. These payout functions define the derivative. The terms are expressed as follows:

Let $W(t)$

$=$ price of a derivative security at time $t$.

(2)    If $V(t) \geqq \bar{V}(t)$ for $0 \leqq t < T$,

then $W(t) = f[V(t), t]$;

If $V(t) \leqq \underline{V}(t)$ for $0 \leqq t < T$,

then $W(t) = g[V(t), t]$;

If $t = T$,    then $W(T) = h[V(T)]$.

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eling instead of the Itô calculus. My report showed that while the paper's mathematics was correct, its economics was not: A Stratonovich-type formulation of the underlying price process implies that traders have a partial knowledge about future asset prices that the nonanticipating character of the Itô process does not. The "paradox" is thus resolved because the assumed information sets are essentially different and hence, so should the pricing formulas.

[7] Since a discontinuous sample-path price process for the underlying asset rules out perfect hedging even with continuous trading but a continuous-sample-path process with stochastic volatility does not, there is considerable interest in testing which process fits the data better. See Eric Rosenfeld (1980), an early developer of such tests, and James B. Wiggins (1987).

[8] The important Douglas T. Breeden (1979) Consumption-based Capital Asset Pricing Model, which was not published at the time of these papers, can also be used to complete those models.

[9] See Black (1989) and Scholes (1998). Fischer Black always maintained with me that the CAPM-version of the option-model derivation was more robust because continuous trading is not feasible and there are transactions costs. As noted in Merton (1973a p. 161), the discrete-time Samuelson-Merton (1969) model also gives the Black-Scholes formula under special conditions.

For $0 \le t \le T$, the derivative security receives a payment flow rate specified by $D_2(V, t)$. The terms as described in (2) are to be interpreted as follows: the first time that $V(t) \ge \bar{V}(t)$ or $V(t) \le \underline{V}(t)$, the owner of the derivative must exchange it for cash according to the schedule in (2). If no such events occur for $t < T$, then the security is redeemed at $t = T$ for cash according to (2). $T$ is called the *maturity date* (or expiration date, or redemption date) of the derivative. The derivative security is thus defined by specifying the contingent payoff functions $f$, $g$, $h$, $D_2$, and $T$. In some cases, the schedules or the boundaries $\bar{V}(t)$ and $\underline{V}(t)$ are contractually specified; in others, they are determined endogenously as part of the valuation process, as in the case of the early-exercise boundary for an American-type option.

By arbitrage restrictions, the derivative security will have *limited liability* if and only if $g \ge 0$, $h \ge 0$, $f \ge 0$, and $D_2(0, t) = 0$.

If (as drawn in Figure 1) the boundaries $\underline{V}(t)$, and $\bar{V}(t)$ are continuous functions, then because $V(t)$ has a continuous sample path in $t$ by Assumption 2, one has that: (i) if $V(t) < \underline{V}(t)$ for some $t$, then there is a $\underline{t}$, $\underline{t} < t$, so that $V(\underline{t}) = \underline{V}(\underline{t})$; and (ii) if $V(t) > \bar{V}(t)$ for some $t$, then there is a $\bar{t}$, $\bar{t} < t$, so that $V(\bar{t}) = \bar{V}(\bar{t})$. Hence, in this case, the inequalities for $V$ can be neglected in (2) and the only relevant region for analysis is $\underline{V}(t) \le V(t) \le \bar{V}(t)$, $0 \le t \le T$.

With the derivative-security characteristics fully specified, we turn now to the fundamental production technology for hedging the risk of issuing a derivative security and for evaluating the cost of its production. To locate the derivation in a more substantive framework, I posit a hypothetical financial intermediary that creates derivative securities in principal transactions for its customers by selling them contracts which are its obligation. It uses the capital markets or transactions with other institutions to hedge the contractual liabilities so created by dynamically trading in the underlying securities following a strategy designed to reproduce the cash flows of the issued contracts as accurately as it can. If the intermediary cannot perfectly replicate the payoffs to the issued derivative, it either obtains adequate equity to bear the residual risks of its imper-
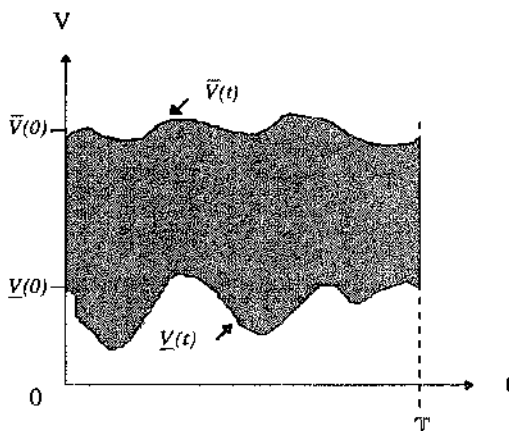


FIGURE 1. RELEVANT REGION OF $V$:
$\underline{V}(t) \le V(t) \le \bar{V}(t)$, $0 \le t \le T$

fectly hedged positions or it securitizes those positions by bundling them into a portfolio for a special-purpose financial vehicle which it then sells either in the capital market or to a consortium of other institutions in a process similar to the traditional reinsurance market. Although surely a caricature, the following description is nevertheless not far removed from real-world practice.

The objective is to find a feasible, continuous-trading portfolio strategy constructed from all available traded assets including the riskless asset that comes "closest" to satisfying the following four properties: if $P(t)$ denotes the value of the portfolio at time $t$, then for $0 \le t \le T$:

(i)  at $t$, if $V(t) = \underline{V}(t)$, then $P(t) = g[\underline{V}(t), t]$;
(ii) at $t$, if $V(t) = \bar{V}(t)$, then $P(t) = f[\bar{V}(t), t]$;
(iii) for each $t$, the payout rate on the portfolio is $D_2(V, t)dt$;
(iv) at $t = T$, $P(T) = h[V(T)]$.

Call this portfolio the "hedging portfolio" for the derivative security defined by (2). That portfolio is labelled as "portfolio (*)." In the special, but important, case in which the portfolio meets the above conditions exactly, the hedging portfolio is called the "replicating portfolio" for the derivative security.

Bertsimas et al. (1997) study the complementary problem of "closeness" of dynamic replication where they assume that one can trade in the underlying asset but that trading is not continuous. They apply stochastic dynamic programming to derive optimal strategies to minimize mean-squared tracking error. These strategies are then employed in simulations to estimate quantitatively how close one can get to dynamic completeness.

I determine the optimal hedging portfolio in two steps: first, find the portfolio strategy constructed from all continuously traded assets that has the smallest "tracking error" in replicating the returns on the underlying asset. For the underlying asset with price $V$, I call this portfolio, the "$V$-Fund." In the second step, derive the hedging portfolio for the derivative security as a dynamic portfolio strategy mixing the $V$-Fund with the riskless asset.

Let $S_i(t)$ denote the price of continuously traded asset $i$ at time $t$. There are $n$ such risky assets plus the riskless asset which are traded continuously. The dynamics for $S_i$ are assumed to follow a continuous-sample-path Itô process given by

$$(3)\quad dS_i = \alpha_i S_i\, dt + \sigma_i S_i\, dZ_i, \quad i = 1, \dots, n,$$

where $\alpha_i$ is the instantaneous expected rate of return on asset $i$; $dZ_i$ is a Wiener process; $\sigma_{ij}$ is the instantaneous covariance between the returns on $i$ and $j$ [that is, $(dS_i/S_i)(dS_j/S_j) = \sigma_{ij}dt$ and $\sigma_{ii} = \sigma_i^2$]; let $\eta_i$ be defined as the instantaneous correlation between $dZ_i$ and $dZ$ in Assumption 2 such that $dZ_i\, dZ = \eta_i\, dt$. Let $S(t)$ denote the value of the $V$-Fund portfolio and let $w_i(t)$ denote the fraction of that portfolio allocated to asset $i$, $i = 1, \dots, n$, at time $t$. The balance of the portfolio's assets are invested in the riskless asset. The dynamics for $S$ can be written as

$$(4)\quad dS = [\mu S - D_1(V, t)]\, dt + \delta S dq,$$

where $\mu = r + \sum_{i=1}^{n} w_i(t)[\alpha_i - r]$, $\delta^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i(t)w_j(t)\sigma_{ij}$, and $dq = [\sum_{i=1}^{n} w_i(t)\sigma_i\, dZ_i]/\delta$.

To create the $V$-Fund, the $w_i$ are chosen so as to minimize the unanticipated part of

the difference between the return on the underlying asset and the traded portfolio's return. That is, at each point in time, the portfolio allocation is chosen so as to minimize the instantaneous variance of $[dS/S - dV/V]$. As shown in Merton (1992, Theorem 15.3 p. 501), the portfolio rule that does this is given by

$$(5)\quad w_i(t) = \sigma \sum_{k=1}^{n} v_{ki}\sigma_k\eta_k, \quad i = 1, \dots, n,$$

where $v_{ki}$ is the $k$th-$i$th element of the inverse of the variance-covariance matrix of the returns on the $n$ risky continuously traded assets. From Merton (1992 p. 502), the instantaneous correlation between the returns on the $V$-Fund and the underlying asset, $\rho\, dt = dZ\, dq$, can be written as

$$(6)\quad \rho = \left( \sum_{k=1}^{n} \sum_{j=1}^{n} v_{ki}\sigma_k\sigma_i\eta_k\eta_i \right)^{1/2}$$

and

$$(7)\quad \delta = \rho\sigma.$$

The dynamics of the tracking error can thus be written as

$$(8)\quad dS/S - dV/V = (\mu - \alpha)\, dt + \theta db,$$

where $\theta^2 = (1 - \rho^2)\sigma^2$ and the Wiener process $db = (\rho\, dq - dZ)/\sqrt{1 - \rho^2}$. As shown in Merton (1992 equation 15.51), it follows that

$$(9)\quad dS_i/S_i\, db = 0, \quad i = 1, \dots, n.$$

That is, the tracking error in (8) is uncorrelated with the returns on all traded assets, which is a consequence of picking the portfolio strategy that minimizes that error.

With this, we now proceed with a "cookbook-like" derivation of the production process for our hypothetical financial intermediary to best hedge the cash flows of the derivative securities it issues. The derivation begins with a description of the activities for the intermediary's quantitative-analysis ("quant") department which is responsible for gathering the variance-covariance information necessary to use (5) to construct and

maintain the $V$-Fund portfolio. It is also assigned the responsibility to solve the following linear parabolic partial differential equation for $F[V, t]$:

$$(10) \quad 0 = \tfrac{1}{2}\sigma^2(V, t)V^2 F_{11}[V, t]$$

$$+ [rV - D_1(V, t)] F_1[V, t]$$

$$- rF[V, t] + F_2[V, t] + D_2(V, t)$$

subject to the boundary conditions: for $\underline{V}(t) \leq V \leq \bar{V}(t)$ and $t < T$,

$$(11) \quad F[\bar{V}(t), t] = f[\bar{V}(t), t] \geq 0;$$

$$(12) \quad F[\underline{V}(t), t] = g[\underline{V}(t), t] \geq 0;$$

$$(13) \quad F[V, T] = h[V] \geq 0,$$

where $F_{11} \equiv \partial^2 F/\partial V^2$, $F_1 \equiv \partial F/\partial V$; and $F_2 \equiv \partial F/\partial t$. Note that the nonnegativity conditions in (11)–(13) together with $D_2(0, t) = 0$ implies that the derivative security has limited liability. As a mathematical question, this is a well-posed problem, and a solution to (10)–(13) exists and is unique.

Having solved for the function $F[V, t]$, the quant department has the prescribed ongoing tasks at each time $t$ ($0 \leq t \leq T$) to:

(i) ask the trading desk for the prices of all traded assets necessary to determine the price $S(t)$ of the $V$-Fund and the best estimate of the current price of the underlying asset, $V(t)$;

(ii) compute from the solution to (10)–(13)

$$M(t) \equiv F_1[V(t), t] V(t);$$

(iii) tell the trading desk that the strategy of portfolio (*) requires that $\$M(t)$ be invested in the $V$-Fund for the period $t$ to $t + dt$;

(iv) compute $Y(t) \equiv F[V(t), t]$ and store $Y(t)$ in the intermediary's data files for (later) analysis of the time series (i.e., stochastic process) $Y(t)$.

The prescription for the execution or trading-desk activities of the intermediary is as follows: At time $t = 0$, give the trading desk $\$P(0)$ as an initial funding (investment) for portfolio (*) which contains the $V$-Fund asset and the riskless asset. Let $P(t)$ denote the value of portfolio (*) at $t$, after having made any prescribed cash distribution (payment) from the portfolio. The trading desk has the job at each time $t$ ($0 \leq t \leq T$) to:

(a) determine the current prices of the underlying asset, $V(t)$ and all individual traded assets held in the $V$-Fund, and send that price information to the quant department;

(b) pay a cash distribution of $\$D_2[V(t), t]dt$ to the customer holding the derivative security, by selling securities in the portfolio (if necessary);

(c) compute the value of the balance of the portfolio, $P(t)$;

(d) receive instructions on $M(t)$ from the quant department;

(e) readjust the portfolio allocation so that $\$M(t)$ is now invested in the $V$-Fund and $\$[P(t) - M(t)]$ is invested in the riskless asset.

It follows that the dynamics for the value of portfolio (*) are given by

$$(14) \quad dP = M(t)\frac{dS}{S} + M(t)\frac{D_1(V, t)}{S}dt$$

$$+ [P - M(t)]rdt - D_2(V, t)dt$$

where

$$M(t)\frac{dS}{S} = \text{price appreciation};$$

$$M(t)\frac{D_1(V, t)}{S}dt = \text{dividend payments}$$

received into the portfolio;

$$[P - M(t)]rdt$$

$$= \text{interest earned by the portfolio};$$

$$D_2(V, t)dt = \text{cash distribution to customer}.$$

Noting that $M(t) = F_1[V, t]V$, one has by substitution from (4) into (14) that the dynamics of $P$ satisfy

$$(15) \quad dP = F_1[V, t]V dS/S$$

$$+ F_1[V, t]V D_1(V, t)/S$$

$$+ (P - F_1[V, t]V) r dt$$

$$- D_2(V) dt$$

$$= [F_1 V(\mu - r) + rP - D_2] dt$$

$$+ F_1 V \delta dq.$$

Return now to the quant department to derive the dynamics for $Y(t)$. From (iv), one has that $Y(t) = F[V, t]$ for $V(t) = V$. Because $F$ is the solution to (10)–(13), $F$ is a twice-continuously differentiable function of $V$ and $t$. Therefore, we can apply Itô's lemma, so that for $V(t) = V$,

$$(16) \quad dY = F_1[V, t]dV + F_2[V, t]dt$$

$$+ \tfrac{1}{2} F_{11}[V, t](dV)^2$$

$$= [\tfrac{1}{2}\sigma^2 V^2 F_{11} + F_1(\alpha V - D_1)$$

$$+ F_2] dt + F_1 V \sigma dZ$$

because $(dV)^2 = \sigma^2 V^2 dt$. Because $F[V, t]$ satisfies (10), one has that

$$(17) \quad \tfrac{1}{2}\sigma^2 V^2 F_{11} - D_1 F_1 + F_2$$

$$= rF - rVF_1 - D_2.$$

Substituting (17) into (16), one can rewrite (16) as

$$(18) \quad dY = [F_1(\alpha - r)V + F - D_2] dt$$

$$+ F_1 V \sigma dZ.$$

Note that the calculation of $Y(t)$ and its dynamics by the quant department in no way requires knowledge of the time series of values for portfolio (*), $\{P(t)\}$, that are calculated

by the trading desk. Putting these two time series together, we define $Q(t) \equiv P(t) - Y(t)$. It follows that $dQ = dP - dY$. Substituting for $dP$ from (15) and for $dY$ from (18), rearranging terms using (8), one has that

$$(19) \quad dQ = rQ dt + F_1 V(dS/S - dV/V)$$

$$= (rQ + F_1 V[\mu - \alpha]) dt$$

$$+ F_1 V \theta db.$$

At this point, I digress to examine the special case in which perfect replication of the return on the underlying asset obtains (i.e., $\rho = 1$ and there is no tracking error). In that case, equation (19) reduces to an ordinary differential equation ($\dot{Q}/Q = r$) with solution

$$(20) \quad Q(t) = Q(0)\exp(rt)$$

where $Q(0) = P(0) - Y(0) = P(0) - F[V(0), 0]$. Therefore, if the initial funding provided to the trading desk for portfolio (*) is chosen so that $P(0) = F[V(0), 0]$, then from (20), $Q(t) = 0$ for all $t$ and

$$(21) \quad P(t) = F[V(t), t].$$

By comparison of (11)–(13) with (2), one has from (21) that the (*)-portfolio strategy generates the identical payment flows and terminal (and boundary) values as the derivative security described at the outset of this analysis. That is, for a one-time, initial investment of $\$F[V(0), 0]$, a feasible portfolio strategy has been found that exactly replicates the payoffs to the derivative security. Thus, $\$F[V(0), 0]$ is the cost to the intermediary for producing the derivative. *If the derivative security is traded,* then to avoid ("conditional") arbitrage (conditional on $\sigma, r, D_1$), its price must satisfy

$$(22) \quad W(t) = P(t) = F[V(t), t].$$

Since the absence of arbitrage opportunities is a *necessary* condition for equilibrium, it follows that equilibrium prices for derivative securities on continuously tradable underlying assets must satisfy (22). This is, of course, the original Black-Scholes result and the $V$-Fund

degenerates into a single asset, the underlying asset itself. However, note that (22) obtains without assuming that the derivative-pricing function is a twice-continuously differentiable function of $V$ and $t$. The smoothness of the pricing function is instead a derived conclusion.

Note further that the development of the (\*)-portfolio strategy did not require that the derivative security [defined by (2)] actually trades in the capital market. The (\*)-portfolio strategy provides the technology for "manufacturing" or synthetically creating the cash flows and payoffs of the derivative security if it does not exist. That is, if one describes a state-contingent schedule of outcomes for a portfolio [i.e., specifies $f$, $g$, $h$, $D_2$, $T$, $\underline{V}(t)$, $\overline{V}(t)$], then the (\*)-portfolio strategy provides the trading rules to create this pattern of payouts and it specifies the cost of implementing those rules. The cost of creating the security at time $t$ is thus $F[V(t), t]$. Moreover, if the financial services industry is competitive, then price equals marginal cost, and (22) obtains as the formula for equilibrium prices of derivatives sold directly by intermediaries.

Returning from this digression to the case of imperfect replication, one has, by construction of the process for $Y$, that $Q = P - Y$ is the cumulative arithmetic tracking error for the hedging portfolio. By inspection of (19), the instantaneous tracking error for the derivative security is perfectly correlated with the tracking error of the $V$-Fund. Hence, from (9), it follows that the tracking error for the hedging portfolio is uncorrelated with the returns on all continuously traded assets. Using this lack of correlation with any other traded asset, I now argue that in this case the replication-based valuation can be used for pricing the derivative security even though replication is not feasible.

As we know, in all equilibrium asset-pricing models, assets that have only nonsystematic or diversifiable risk are priced to yield an expected return equal to the riskless rate of interest. The condition satisfied by the tracking-error component of the hedging portfolio satisfies an even stronger no-correlation condition than either a zero-beta asset in the CAPM, a zero multibeta asset of the Intertemporal CAPM, or a zero factor-risk asset of the

Arbitrage Pricing Theory. Thus, by any of those theories, the equilibrium condition from either (8) or (19) is that

$$(23) \qquad \mu = \alpha.$$

If (23) obtains, it follows immediately that the equilibrium price for the derivative security is $F[V(t), t]$, the same formula "as if" the underlying asset is traded continuously. And as a consequence, the Black-Scholes formula would apply even in those applications in which the underlying asset is not traded.

As is well known from the literature on incomplete markets, (23) need not obtain if the creation of the new derivative security helps complete the market for a large enough subset of investors that the incremental dimension of risk spanned by this new instrument is "priced" as a systematic risk factor with an expected return different from the riskless interest rate. Markets tend to remain incomplete with respect to a particular risk either because the cost of creating the securities necessary to span that risk exceeds the benefits, or because nonverifiability, moral-hazard, or adverse-selection problems render the viability of such securities untenable. Generally, major macro risks for which significant pools of investors want to manage their exposures are not controllable by any group of investors, and it is unlikely that any group would have systematic access to materially better information about those risks. Hence, the usual asymmetric-information and incentive reasons given for market failure do not seem to be present. In systems with well-developed financial institutions and markets and with today's financial technology, it is thus not readily apparent what factors make the cost of developing standardized derivative markets (e.g., futures, swaps, options) prohibitive if, in large scale, there is a significant premium latently waiting to be paid by investors who currently participate in the markets. On a more prosaic empirical note, in most applications of the option-pricing model, the "residual" or tracking-error variations are likely to be specific to the underlying project, firm, institution, or person, and thereby they are unlikely candidates for macro-risk surrogates. These observations support the prospects for (23) to obtain.

However, the risk need not be macro in scope in order to be significant to one investor or a small group of investors. Obvious examples of such risks would be various firm- or person-specific components of human capital, including death and disability risks. To make a case for instruments with these types of exposures to be priced with a risk premium, incomplete-market models often focus on the "incipient-demand" (or "maximum reservation") price or risk premium that an investor would pay to eliminate a risk that is not covered in the market by the existing set of securities. In the abstract, that price, of course, can be quite substantial. However, arguments along these lines to explain financial product pricing implicitly assume a rather modest and static financial services sector. A classic example is life insurance. Risk-averse individuals with families may, if necessary, be willing to pay a considerable premium for life insurance, well in excess of the actuarial mortality risk, even after taking into account moral-hazard and person-specific informational asymmetries. Moreover, if the analysis further postulates a financial sector so crude that bilateral contracts between risk-averse individuals are the only way to obtain such insurance, then the equilibrium price for such insurance in that model can be so large that few, if any, contracts are created. But, such models are a poor descriptor of the real world. If the institutions and markets were really that limited, the incentives for change and innovation would be enormous. Modern finance technology and experience in implementing it provide the means for such change. And if, instead, one admits into the model just the classic mechanism for organizing an "insurance" institution (whether government-run or private sector) to take advantage of the enormous diversification benefits of pooling such risks and subdividing them among large numbers of participants, then the equilibrium price equals the "supply" price of such insurance contracts which approaches the actuarial rate.

As is typical in analyses of other industries, the equilibrium prices of financial products and services are more closely linked to the costs of the efficient producers than to the inefficient ones (except perhaps as a very crude upper bound to those prices). Furthermore, the institutional structure of the financial system is neither exogenous nor fixed. In theory and in practice, that structure changes in response to changing technology and to profit opportunities for creating new products and existing products more efficiently. As discussed at length elsewhere (Merton, 1992 pp. 457–67, 535–36), a financial sector with a rich and well-developed structure of institutions can justify a "quasi-dichotomy" modeling approach to the pricing of real and financial assets that employs "reduced-form" equilibrium models with a simple financial sector in which all agents are assumed to be minimum-cost information processors and transactors. However, distortions of insights into the real world can occur if significant costs for the agents are introduced into the model while the simple financial sector is retained as an unchanged assumption. Put simply, high transaction and information costs for most of the economy's agents to directly create their own financial products and services does not imply that equilibrium asset prices are influenced by those high costs, as long as there is an efficient financial service industry with low-cost, reasonably competitive producers.

In considering the preceding technical analysis, one might wonder if there are relevant situations in which the price is observable but trade in the asset cannot take place? One common class of real-world instances is characterized as follows: consider an insurance company that has guaranteed the financial performance of the liabilities of a privately held opaque institution with a mark-to-market portfolio of assets. The market value of that portfolio (corresponding to $V$ in the analysis here) is provided to the guarantor on a continuous basis, but the portfolio itself cannot be traded by the guarantor to hedge its exposure because it does not know the assets held within the portfolio. Elsewhere (Merton, 1997a), I have developed a model using an alternative approach of incentive-contracting combined with the derivative-security technology to analyze the problem of contract guarantees for an opaque institution. It is nevertheless the case that discontinuous tradability of an asset is often accompanied by discontinuous observations of its price. And so, the combination of the two warrants attention. Hence, I com-

plete this section with consideration of how to modify the valuation formula if the price of the underlying asset $V$ is not continuously observable.

Suppose that in the example adopted in this section, the price of the underlying asset is observed at $t = 0$ and then again at the maturity of the derivative contract, $t = T$. In between, there is neither direct observation nor inferential information from payouts on the asset. Hence, $D_1(V, t) = 0$, and the derivative security has no payouts or interim "stopping points" prior to maturity [as specified in (11) and (12)] contingent on $V(t)$. It is however known that the dynamics of $V$ are as described in Assumption 2 with a covariance structure with available traded assets sufficiently well specified to construct the $V$-Fund according to (5). Define the random variable $X(t) \equiv V(t)/S(t)$, the cumulative *proportional* tracking error, with $X(0) = 1$. By applying Ito's lemma, one has from (8), (9), and (23) that the dynamics for $X$ can be written as

$$(24) \qquad dX = \theta X db.$$

It follows from (24) that the distribution for $X(t)$, conditional on $X(0) = 1$, is lognormal with the expected value of $X(t)$ equal to 1 and the variance of $\ln[X(t)]$ equal to $\theta^2 t$. The partial differential equation for $F$, corresponding to (10), that determines the hedging strategy uses as its independent variable the best estimate of $V(t)$, which is $S(t)$, and it is written as

$$(25) \quad 0 = \tfrac{1}{2}\delta^2 S^2 F_{11}[S, t] + rSF_1[S, t]$$
$$- rF[S, t] + F_2[S, t],$$

subject to the terminal-time boundary condition that for $S(T-) = S$,

$$(26) \qquad F[S, T] = E\{h(SX)\},$$

where $h$ is as defined in (13), $X$ is a lognormally distributed random variable with $E\{X\} = 1$ and variance of $\ln[X]$ equal to $\theta^2 T$ and $E\{\ \}$ is the expectation operator over the distribution of $X$.

Condition (26) reflects the fact that for all $t < T$, the best estimate of $V(t)$ is $S(t)$. How-

ever, at $t = T$, $V(T)$ is revealed and the value of $S$ "jumps" by the total cumulative tracking error of $X(T)$ from its value $S$ at $t = T-$ to $S(T) = V(T)$. The effect of the underlying asset price not being observable is perhaps well illustrated by comparing the solution for the European-type call option with the classic Black-Scholes solution given here in (1). The solution to (25) and (26) with $h(V) = \max[0, V - L]$ is given by, for $0 < t < T$,

$$(27) \quad F[S, t]\}$$
$$= SN(u) - L\exp(-r[T - t])$$
$$\times N(u - \sqrt{\gamma}),$$

where $u = (\ln[S/L] + r[T - t] + \gamma/2)/\sqrt{\gamma}$, $\gamma = \delta^2(T - t) + \theta^2 T$, and $N(\ )$ is the cumulative density function for the standard normal distribution.

By inspection of (1) and (27), the key difference in the option-pricing formula with and without continuous observation of the underlying asset price is that the variance over the remaining life of the option does not go to zero as $t$ approaches $T$, because of the "jump" event at the expiration date corresponding to the cumulative effect of tracking error.

This section has explored conditions under which the Black-Scholes option-pricing model can be validly applied to the pricing of assets with derivative-security-like structures, even when the underlying asset-equivalent is neither continuously traded nor continuously observable. A fuller analysis of this question would certainly take account of the additional tracking error that obtains as a consequence of imperfect dynamic trading of the $V$-Fund portfolio, along the lines of Bertsimas et al. (1997). However, a more accurate assessment of the real-world impact should also take into account other risk-management tools that intermediaries have to reduce tracking error. For instance, as developed in analytical detail in Merton (1992 pp. 450–57), intermediaries need only use dynamic trading to hedge their *net* derivative-security exposures to various underlying assets. For a real-world intermediary with a large book of various derivative products, netting, which in effect extends the capability for hedging to include trading in

securities with "nonlinear" payoff structures, can vastly reduce the size and even the frequency of the hedging transactions necessary to achieve an acceptable level of tracking error. Beyond this, as part of their optimal risk management, intermediaries can "shade" their bid and offer prices among their various products to encourage more or less customer activity in different products to help manage their exposures. The limiting case when the net positions of customer exposures leaves the intermediary with no exposure is called a "matched book."

## II. Applications of the Option-Pricing Technology

Open the financial section of a major newspaper almost anywhere in the world and you will find pages devoted to reporting the prices of exchange-traded derivative securities, both futures and options. Along with the vast over-the-counter derivatives market, these exchange markets trade options and futures on individual stocks, stock-index and mutual-fund portfolios, on bonds and other fixed-income securities of every maturity, on currencies, and on commodities including agricultural products, metals, crude oil and refined products, natural gas, and even, electricity. The volume of transactions in these markets is often many times larger than the volume in the underlying cash-market assets. Options have traditionally been used in the purchase of real estate and the acquisition of publishing and movie rights. Employee stock options have long been granted to key employees and today represent a significantly growing proportion of total compensation, especially for the more highly paid workers in the United States. In all these markets, the same option-pricing methodology set forth in the preceding section is widely used both to price and to measure the risk exposure from these derivatives (cf., Robert A. Jarrow and Andrew T. Rudd [1983] and Cox and Mark Rubinstein [1985]). However, financial options represent only one of several categories of applications for the option-pricing technology.

In the late 1960's and early 1970's when the basic research leading to the Black-Scholes

model was underway, options were seen as rather arcane and specialized financial instruments. However, both Black and Scholes (1972, 1973) and I (Merton [1970, 1974]) recognized early on in the research effort that the same approach used to price options could be applied to a variety of other valuation problems. Perhaps the first major development of this sort was the pricing of corporate liabilities, the "right-hand side" of the firm's balance sheet. This approach to valuation treated the wide array of instruments used to finance firms such as debentures, convertible bonds, warrants, preferred stock, and common stock (as well as a variety of hybrid securities) as derivative securities with their contractual payouts ultimately dependent on the value of the overall firm. In contrast to the standard fragmented valuation methods of the time, it provided a unified theory for pricing these liabilities. Because application of the pricing methodology does not require a history of trading in the particular instrument to be evaluated, it was well suited for pricing new types of financial securities issued by corporations in an innovating environment. Applications to corporate finance along this line developed rapidly.[10]

"Option-like" structures were soon seen to be lurking everywhere; thus there came an explosion of research in applying option-pricing which still continues. Indeed, I could not do full justice to the list of contributions accumulated over the past 25 years even if this entire paper were devoted to that endeavor. Fortunately, a major effort to do just that is underway and the results will soon be available (Jin et al., 1998). The authors have generously shared their findings with me. And so, I can convey here some sense of the breadth of applications and be necessarily incomplete without harm.

The put option is a basic option which gives its owner the right to sell the underlying asset at a specified ("exercise") price on or before a given ("expiration") date. When purchased in conjunction with ownership of the under-

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[10] See Merton (1992 pp. 423–27) for an extensive list of references. See also Gregory D. Hawkins (1982) and Michael J. Brennan and Eduardo S. Schwartz (1985a) and the early empirical testing by E. Philip Jones et al. (1984).

lying asset, it is functionally equivalent to an insurance policy that protects its owner against economic loss from a decline in the asset's value below the exercise price for any reason, where the term of the insurance policy corresponds to the expiration date. Hence, option-pricing theory can be applied to value insurance contracts. An early insurance application of the Black-Scholes model was to the pricing of loan guarantees and deposit insurance (cf., Merton, 1977a). A contract that insures against losses in value caused by default on promised payments on a contract in effect is equivalent to a put option on the contract with an exercise price equal to the value of the contract if it were default free. Loan and other contract guarantees, collectively called credit derivatives, are ubiquitous in the private sector. Indeed, whenever a debt instrument is purchased in which there is any chance that the promised payments will not be made, the purchaser is not only lending money but also in effect issuing a loan guarantee as a form of self-insurance. Another private-sector application of options analysis is in the valuation of catastrophic-insurance reinsurance contracts and bonds.[11] Dual funds and exotic options provide various financial-insurance and minimum-return-guarantee products.[12]

Almost surely, the largest issuer of such guarantees are governments. In the United States, the Office of the Management of the Budget is required by law to value those guarantees. The option model has been applied to assess deposit insurance, pension insurance,

guarantees of student loans and home mortgages, and loans to small businesses and some large ones as well.[13] The application to government activities goes beyond just providing guarantees. The model has been used to determine the cost of other subsidies including farm-price supports and through-put guarantees for pipelines.[14] It has been applied to value licenses issued with limiting quotas such as for taxis or fisheries or the right to pollute and to value the government's right to change those quotas.[15] Government sanctions patents. The decision whether to spend the resources to acquire a patent depends on the value of the patent which can be framed as an option-pricing problem. Indeed, even on something that is not currently commercial, one may acquire the patent for its "option value," should economic conditions change in an unexpected way.[16] James L. Paddock et al. (1988) show that option value can be a significant proportion of the total valuation of government-granted offshore drilling rights, especially when current and expected future economic conditions would not support development of the fields. Option-pricing analysis quantifies the government's economic decision whether to build roads in less-populated areas depending on whether it has the policy option to abandon rural roads if they are not used enough.[17]

Various legal and tax issues involving policy and behavior have been addressed using the option model. Among them is the valuation of plaintiffs' litigation options, bankruptcy laws including limited-liability provisions, tax delinquency on real estate and other property as an option to abandon or recover the property by paying the arrears, tax evasion, and valuing the tax "timing" option for the

[11] Cf., Alan Kraus and Ross (1982), Neil A. Doherty and James R. Garven (1986), J. David Cummins (1988), Cummins and Hélyette Geman (1995), and Scott E. Harrington et al. (1995).

[12] Brennan and Schwartz (1976), Jonathan J. Ingersoll, Jr. (1976), M. Barry Goldman et al. (1979), Mary Ann Gatto et al. (1980), and René M. Stulz (1982). In an early real-world application, Myron Scholes and I developed the first options-strategy mutual fund in the United States, Money Market/Options Investments, Inc., in February 1976. The strategy, which invested 90 percent of its assets in money-market instruments and 10 percent in a diversified portfolio of stock call options, provided equity exposure on the upside with a guaranteed "floor" on the value of the portfolio. The return patterns from this and similar "floor" strategies were later published in Merton et al. (1978, 1982).

[13] Howard B. Sosin (1980), Baldwin et al. (1983), Donald F. Cunningham and Patric H. Hendershott (1984), Alan J. Marcus (1987), Merton and Zvi Bodie (1992), Bodie (1996), Ashoka Mody (1996), and Robert S. Neal (1996).

[14] Scott P. Mason and Merton (1985), Calum G. Turvey and Vincent Amanor-Boadu (1989), and Taehoon Kang and B. Wade Brorsen (1995).

[15] James E. Anderson (1987) and Jonathan M. Karpoff (1989).

[16] Lenos Trigeorgis (1993).

[17] Cathy A. Hamlett and C. Phillip Baumel (1990).

capital-gains tax in a circumstance when only realization of losses and gains on investments triggers a taxable event.[18]

In a recent preliminary study, the options structure has been employed to help model the decision of whether the Social Security fund should invest in equities (Kenneth Smetters, 1997). As can be seen in the option formula of the preceding section, the value of an option depends on the volatility of the underlying asset. The Federal Reserve uses as one of its indicators of investor uncertainty about the future course of interest rates, the "implied" volatility derived from option prices on government bonds.[19] In his last paper, published after his death, Black (1995) applies options theory to model the process for the interest rates that govern the dynamics of government bond prices. In another area involving central-bank concerns, Perold (1995) shows how the introduction of various types of derivatives contracts has helped reduce potential systemic-risk problems in the payment system from settlement exposures. The Black-Scholes model can be used to value the "free credit option" implicitly offered to participants, in addition to "float," in markets with other than instantaneous settlement periods. See also Paul H. Kupiec and Patricia A. White (1996). The prospective application of derivative-security technology to enhance central-bank stabilization policies in both interest rates and currencies is discussed in Merton (1995, 1997b).

In an application involving government activities far removed from sophisticated and relatively efficient financial markets, options analysis has been used to provide new insights into optimal government planning policies in developing countries. A view held by some in development economics about the optimal educational policy for less-developed countries is that once the expected future needs for labor-force composition are determined, the optimal education policy should be to pursue targeted training of the specific skills forecast and in the quantities needed. The alternative of providing either more general education and training in multiple skills or training in skills not expected to be used is seen as a "luxury" that poorer, developing countries could not afford. It, of course, was understood, that forecasts of future labor-training needs were not precise. Nevertheless, the basic prescription formally treated them as if they were. In Samantha J. Merton (1992), the question is revisited, this time with an explicit recognition of the uncertainty about future labor requirements embedded in the model. The analysis shows that the value of having the option to change the skill mix and skill type of the labor force over a relatively short period of time can exceed the increased cost in terms of longer education periods or less-deep training in any one skill. The Black-Scholes model is used to quantify that trade-off. In a different context of the private sector in a developed country, the same technique could be used to assess the cost-benefit trade-off for a company to pay a higher wage for a labor force with additional skills not expected to be used in return for the flexibility to employ those skills if the unexpected happens.

The discussion of labor education and training decisions and litigation and taxes leads naturally into the subject of human capital and household decision-making. The individual decision as to how much vocational education to acquire can be formulated as an option-valuation problem in which the optimal exercise conditions reflect when to stop training and start working.[20] In the classic labor-leisure trade-off, one whose job provides the flexibility to increase or decrease the number of hours worked, and hence his total compensation, on relatively short notice, has a valuable option relative to those whose available work hours are fixed.[21] Wage and pension-plan "floors" that provide for a minimum compensation, and even tenure for university professors (John G. McDonald, 1974), have an option-like structure. Other options commonly a part of house-

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[18] George M. Constantinides and Ingersoll (1984), Brendan O'Flaherty (1990), William J. Blanton (1995), Paul G. Mahoney (1995), and Charles T. Terry (1995).

[19] Sylvia Nasar (1992). See Bodie and Merton (1995) for an overview article on implied volatility as an example of the informational role of asset and option prices.

[20] Uri Dothan and Joseph Williams (1981).

[21] Bodie et al. (1992).

hold finance are: the commitment by an institution to provide a mortgage to the house buyer, if he chooses to get one; the prepayment right, after he takes the mortgage, that gives the homeowner the right to renegotiate the interest rate paid to the lender if rates fall;[22] a car lease which gives the customer the right, but not the obligation, to purchase the car at a prespecified price at the end of the lease.[23] Health-care insurance contains varying degrees of flexibility, a major one being whether the consumer agrees in advance to use only a prespecified set of doctors and hospitals ("HMO plan") or he retains the right to choose an "out-of-plan" doctor or hospital ("point-of-service" plan). In the consumer making the decision on which to take and the health insurer assessing the relative cost of providing the two plans, each solves an option-pricing problem as to the value of that flexibility.[24] Much the same structure of valuation occurs in choosing between "pay-per-view" and "flat-fee" payment for cable-television services.

Many of the preceding option-pricing applications do not involve financial instruments. The family of such applications is called "real" options. The most developed area for real-option application is investment decisions by firms.[25] However, real-options analysis has also been applied to real-estate investment and development decisions.[26] The common element for using option-pricing here is the same as in the preceding examples: the future is uncertain (if it were not, there would be no need to create options because we know now what we will do later) and in an uncertain environment, having the flexibility to decide what to do after some of that uncertainty is resolved definitely has value. Option-pricing theory provides the means for assessing that value.

The major categories of options within project-investment valuations are: the option to initiate or expand; the option to abandon or contract; and the option to wait, slow down, or speed up development. There are "growth" options which involve creating excess capacity as an option to expand and research and development as creating the opportunity to produce new products and even new businesses, but not the obligation to do so if they are not economically viable.[27]

A few examples: For real-world application of the options technology in valuing product development in the pharmaceutical industry, see Nichols (1994). In the generation of electric power, the power plant can be constructed to use a single fuel such as oil or natural gas or it can be built to operate on either. The value of that option is the ability to use the least-cost, available fuel at each point in time and the cost of that optionality is manifest in both the higher cost of construction and less-efficient energy conversion than with the corresponding specialized equipment. A third example described in Timothy A. Luehrman (1992) comes from the entertainment industry and involves the decision about making a sequel to a movie; the choices are: either to produce both the original movie and its sequel at the same time, or wait and produce the sequel after the success or failure of the original is known. One does not have to be a movie-production expert to guess that the incremental cost of producing the sequel is going to be less if the first path is followed. While this is done, more typically the latter is chosen, especially with higher-budget films. The economic reason is that the second approach provides the option not to make the sequel (if, for example, the original is not a success). If the producer knew (almost certainly) that the sequel will be produced, then the option value of waiting for more information is small and the cost of

[22] Kenneth B. Dunn and John J. McConnell (1981) and Brennan and Schwartz (1985b).

[23] Stephen E. Miller (1995).

[24] James A. Hayes et al. (1993) and Frank T. Magiera and Robert A. McLean (1996).

[25] Mason and Merton (1985), Robert L. McDonald and Daniel R. Siegel (1985), Saman Majd and Robert S. Pindyck (1987), Alexander J. Triantis and James E. Hodder (1990), Avinash K. Dixit and Pindyck (1994), Nancy A. Nichols (1994), Trigeorgis (1996), and Keith J. Leslie and Max P. Michaels (1997).

[26] V. Kerry Smith (1984), Raymond Chiang et al. (1986), David Geltner and William C. Wheaton (1989), Joseph T. Williams (1991), and F. Christian Zinkhan (1991).

[27] W. Carl Kester (1984), Robyn McLaughlin and Robert A. Taggart (1992), and Terrance W. Faulkner (1996).

doing the sequel separately is likely to exceed the benefit. Hence, once again, we see that the amount of uncertainty is critical to the decision, and the option-pricing model provides the means for quantifying the cost/benefit trade-off. As a last example, Baldwin and Clark (1999) develop a model for designing complex production systems focused around the concept of modularity. They exemplify their central theme with several industrial examples which include computer and automobile production. Modularity in production provides options. In assessing the value of modularity for production, they employ an option-pricing type of methodology, where complexity in the production system is comparable to uncertainty in the financial one.[28]

In each of these real-option examples as with a number of the other applications discussed in this section, the underlying ''asset'' is rarely traded in anything approximating a continuous market and its price is therefore not continuously observable either. For that reason, this paper, manifestly focused on applications, devotes so much space to the technical section on extending the Black-Scholes option-pricing framework to include nontradability and nonobservability.

### III. Future Directions of Applications

As I suggested at the outset, innovation is a central force driving the financial system toward greater economic efficiency with considerable economic benefit having accrued from the changes since the time that the option-pricing papers were published. Indeed, much financial research and broad-based practitioner experience developed over that period have led to vast improvements in our understanding of how to apply the new financial technologies to manage risk. Moreover, we have seen how

wide ranging are the applications of our technology for pricing and measuring the risk of derivatives. Nevertheless, there still remains an intense uneasiness among managers, regulators, politicians, the press, and the public over these new derivative-security activities and their perceived risks to financial institutions. And this seems to be the case even though the huge financial disruptions, such as the savings-and-loan debacle of the 1980's in the United States and the current financial crises in Asia and some emerging markets, appear to be the consequence of the more traditional risks taken by institutions such as commercial, real-estate, and less-developed-country lending, loan guarantees, and equity-share holdings.

One conjecture attributes this uneasiness to the frequently cited instances of individual costly events that are alleged to be associated with derivatives, such as the failure of Barings Bank, Proctor and Gamble's losses on complex interest rate contracts, the financial distress of Orange County, and so forth. Perhaps.[29] But, as already noted, derivatives are ubiquitous in the financial world and thus, they are likely to be present in any financial circumstance, whether or not their use has anything causal to do with the resulting financial outcomes. However, even if all these allegations were valid, the sheer fact that we are able to associate individual names with these occurrences instead of mere numbers (''XYZ company'' instead of ''475–500 thrifts'' as the relevant descriptor) would suggest that these are relatively isolated events—unfortunate pathologies rather than indicators of systemic flaws. In contrast, the physiology of this financial technology, that is, how it works when it works as it should, is not the subject of daily reports from around the globe but is essentially taken for granted.

An alternative or supplementary conjecture about the sources of the collective anxiety over derivatives holds that they are a part of a wider implementation of financial innovations which have required major changes in the basic in-

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[28] See also Hua He and Pindyck (1992). On an entirely different application, Kester's (1984) analysis of whether to develop products in parallel or sequentially could be applied to the evaluation of alternative strategies for funding basic scientific research: is it better to support *N* different research approaches simultaneously or just to support one or two and then use the resulting outcomes to sequence future research approaches? See also Merton (1992 p. 426).

[29] Merton H. Miller (1997) provides a cogent analysis refuting many of the specific-case allegations of derivatives misuse.

stitutional hierarchy and in the infrastructure to support it. As a result, the knowledge base now required to manage and oversee financial institutions differs greatly from the traditional training and experience of many financial managers and government regulators. Experiential changes of this sort are threatening. It is difficult to deal with change that is exogenous to our traditional knowledge base and framework and thus comes to seem beyond our control. Decreased understanding of the new environment can create a sense of greater risk even when the objective level of risk in the system remains unchanged or is actually reduced. If so, we should start to deal with the problem now since the knowledge gap may widen if the current pace of financial innovation, as some anticipate, accelerates into the twenty-first century. Moreover, greater complexity of products and the need for more rapid decision-making will probably increase the reliance on models, which in turn implies a growing place for elements of mathematical and computational maturity in the knowledge base of managers. Dealing with this knowledge gap offers considerable challenge to private institutions and government as well as considerable opportunity to schools of management and engineering and to university departments of economics and mathematics.

There are two essentially different frames of reference for trying to analyze and understand changes in the financial system. One perspective takes as given the existing institutional structure of financial service providers, whether governmental or private sector, and examines what can be done to make those institutions perform their particular financial services more efficiently and profitably. An alternative to this traditional institutional perspective—and the one I favor—is the functional perspective, which takes as given the economic functions served by the financial system and examines what is the best institutional structure to perform those functions.[30] The basic functions of a financial system are essentially the same in all economies, which

makes them far more stable, across time and across geopolitical borders, than the identity and structure of the institutions performing them. Thus, a functional perspective offers a more robust frame of reference than an institutional one, especially in a rapidly changing financial environment. It is difficult to use institutions as the conceptual "anchor" for analyzing the evolving financial system when the institutional structure is itself changing significantly, as has been the case for the past two decades and as appears likely to continue well into the future. In contrast, in the functional perspective, institutional change is endogenous, and may therefore prove especially useful in predicting the future direction of financial innovation, changes in financial markets and intermediaries, and regulatory design.[31]

The successful private-sector and governmental financial service providers and overseers in the impending future will be those who can address the disruptive aspects of innovation in financial technology while still fully exploiting its efficiency benefits. What types of research and training will be needed to manage financial institutions? The view of the future here as elsewhere in the economic sphere is clouded with significant uncertainties. With this in mind, I nevertheless try my hand at a few thoughts on the direction of change for product and service demands by users of the financial system and the

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[30] For elaboration on the functional perspective, see Merton (1993, 1995), Dwight B. Crane et al. (1995), and Bodie and Merton (1998).

[31] During the last 25 years, finance theory has been a good predictor of future changes in finance practice. That is, when theory seems to suggest that something "should be there" and it isn't, practice has evolved so that it is. The "pure" securities developed by Kenneth J. Arrow (1953) that so clearly explain the theoretical function of financial instruments in risk bearing were nowhere to be found in the real world until the broad development of the options and derivative-security markets. It is now routine for financial engineers to disaggregate the cash flows of various securities into their elemental Arrow-security component parts and then to reaggregate them to create securities with new patterns of cash flows. For the relation between options and Arrow securities and the application of the Black-Scholes model to the synthesis and pricing of Arrow securities, see Ross (1976b), Rolf W. Banz and Miller (1978), Breeden and Robert H. Litzenberger (1978), Darrell J. Duffie and Chi-fu Huang (1986), and Merton (1992 pp. 443–50).

implications of those changes for applications of mathematical financial modeling.

The household sector of users in the more fully developed financial systems has experienced a secular trend of disaggregation in financial services. Some see this trend continuing with existing products such as mutual funds being transported into technologically less-developed systems. Perhaps so, especially in the more immediate future, with the widespread growth of relatively inexpensive Internet accessibility. However, deep and wide-ranging disaggregation has left households with the responsibility for making important and technically complex micro financial decisions involving risk (such as detailed asset allocation and estimates of the optimal level of life-cycle saving for retirement) —decisions that they had not had to make in the past, are not trained to make in the present, and are unlikely to execute efficiently even with attempts at education in the future. The low-cost availability of the Internet does not solve the "principal-agent" problem with respect to financial advice dispensed by an agent. That is why I believe that the trend will shift toward more integrated financial products and services, which are easier to understand and more tailored toward individual profiles. Those products and services will include not only the traditional attempt to achieve an efficient risk-return trade-off for the tangible-wealth portfolio but will also integrate human-capital considerations, hedging, and income and estate tax planning into the asset-allocation decisions. Beyond the advisory role, financial service providers will undertake a role of principal to create financial instruments that eliminate "short-fall" or "basis" risk for households with respect to targeted financial goals such as tuition for children's higher education and desired consumption-smoothing throughout the life cycle (e.g., preserving the household's standard of living in retirement; cf., Franco Modigliani, 1986). The creation of such customized financial instruments will be made economically feasible by the derivative-security pricing technology that permits the construction of custom products at "assembly-line" levels of cost. Paradoxically, making the products more user-friendly and simpler to under-

stand for customers will create considerably more complexity for the producers of those products. Hence, financial-engineering creativity and the technological and transactional bases to implement that creativity, reliably and cost-effectively, are likely to become a central competitive element in the industry. The resulting complexity will require more elaborate and highly quantitative risk-management systems within financial service firms and a parallel need for more sophisticated approaches to government oversight. Neither of these can be achieved without greater reliance on mathematical financial modeling, which in turn will be feasible only with continued improvements in the sophistication and accuracy of financial models.

Nonfinancial firms currently use derivative securities and other contractual agreements to hedge interest rate, currency, commodity, and even equity price risks. With improved lower-cost technology and learning-curve experience, this practice is likely to expand. Eventually, this alternative to equity capital as a cushion for risk could lead to a major change of corporate structures as more firms use hedging to substitute for equity capital, thereby moving from publicly traded shares to closely held private shares.

The preceding section provides examples of current applications of the options technology to corporate project evaluation: the evaluation of research-and-development projects in pharmaceuticals and the value of flexibility in the decision about sequel production in the movie industry. The big potential shift in the future, however, is from tactical applications of derivatives to strategic ones.[32] For example, a hypothetical oil company with crude oil reserves and gasoline and heating-oil distribution but no refining capability could complete the vertical integration of the firm by using contractual agreements instead of physical acquisition of a refinery. Thus, by entering into contracts that call for the delivery of crude oil by the firm on one date in return for receiving a mix

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[32] See Kester (1984), Stewart C. Myers (1984), and Edward H. Bowman and Dileep Hurry (1993) on the application of option-pricing theory to the evaluation of strategic decisions.

of refined petroleum products at a prespecified later date, the firm in effect creates a synthetic refinery. Real-world strategic examples in natural gas and electricity are described in Harvard Business School case studies, "Enron Gas Services" (1994) and "Tennessee Valley Authority: Option Purchase Agreements" (1996), by Peter Tufano. There is some evidence that these new financial technologies may even lead to a revisiting of the industrial-organization model for these industries.

It is no coincidence that the early strategic applications are in energy- and power-generation industries that need long-term planning horizons and have major fixed-cost components on a large scale with considerable uncertainty. Since energy and power generation are fundamental in every economy, this use for derivatives offers mainline applications in both developed and developing countries. Eventually, such use of derivatives may become standard tools for implementing strategic objectives.

A major requirement for the efficient broad-based application of these contracting technologies in both the household and nonfinancial-firm sectors will be to find effective organizational structures for ensuring contract performance, which includes global clarification and revisions of the treatment of such contractual agreements in bankruptcy. The need for assurances on contract performance is likely to stimulate further development of the financial-guarantee business for financial institutions. Such institutions will have to improve the efficiency of collateral management further as assurance for performance. As we have seen, one early application of the option-pricing model focuses directly on the valuation and risk-exposure measurement of financial guarantees.

A consequence of all this prospective technological change will be the need for greater analytical understanding of valuation and risk management by users, producers, and regulators of derivative securities. Furthermore, improvements in efficiency from derivative products will not be effectively realized without concurrent changes in the financial "infrastructure" — the institutional interfaces between intermediaries and financial markets, regulatory practices, organization of trading,

clearing, settlement, other back-office facilities, and management-information systems. To perform its functions as both user and overseer of the financial system, government will need to innovate and make use of derivative-security technology in the provision of risk-accounting standards, designing monetary and fiscal policies, implementing stabilization programs, and overseeing financial-system regulation.

In summary, in the distant past, applications of mathematical models had only limited and sidestream effects on finance practice. But in the last quarter century since the publication of the Black-Scholes option-pricing theory, such models have become mainstream to practitioners in financial institutions and markets around the world. The option-pricing model has played an active role in that transformation. It is safe to say that mathematical models will play an indispensable role in the functioning of the global financial system.

Even this brief discourse on the application to finance practice of mathematical models in general and the option-pricing model in particular would be negligently incomplete without a strong word of caution about their use. At times we can lose sight of the ultimate purpose of the models when their mathematics become too interesting. The mathematics of financial models can be applied precisely, but the models are not at all precise in their application to the complex real world. Their accuracy as a useful approximation to that world varies significantly across time and place. The models should be applied in practice only tentatively, with careful assessment of their limitations in each application.

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