A RISK MEASURE THAT GOES BEYOND COHERENCE*

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Abstract: There are more to a risk-measure than being coherent. Both the popular VaR and the coherent Tail-VaR ignore useful information in a large part of the loss distribution; As a result they lack incentive for risk-management. I propose a new coherent risk-measure that utilizes information in the whole loss distribution and provides incentive for risk-management.

Keywords: Coherent Risk Measure, VaR, Tail-Var, Distorted Probability.

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1. Introduction

Capital requirement risk-measures are used to decide required capital for a given risk portfolio, based on its downside risk potential. A popular risk-measure for capital requirement in the banking industry is the Value at Risk (VaR), based on a percentile concept. From shareholders' or management's perspective, the quantile "VaR" at the company level is a meaningful risk-measure since the default event itself is of primary concern, and the size of shortfall is only secondary.

From a regulatory perspective, Professors Artzner, Delbaen, Eber, and Heath (1999) advocated a set of consistency rules for a risk-measure. They demonstrated that VaR does not satisfy these consistency rules. Even for shareholders and management, a consistent evaluation of the risks for business units and alternative strategies would require a coherent risk-measure other than VaR.

Artzner et al. (1999) proposed an alternative risk measure --- "Conditional Tail Expectation" (CTE), also called the Tail-VaR[1]. It reflects the mean size of losses exceeding the quantile "VaR", but it ignores losses below the quantile "VaR."

For the sake of portfolio optimization and sound risk-management, it is essential for a risk-measure to properly reflect the risk differentials in alternative strategies or portfolios. Employing a poor risk-measure may have the consequence of making sub-optimal decisions.

In this paper we argue that a risk-measure should go beyond coherence. Although being coherent, Tail-VaR ignores useful information in a large part of the loss distribution, and consequently lacks incentive for mitigating losses below the quantile "VaR". Moreover, Tail-VaR does not properly adjust for extreme low-frequency and high-severity losses, since it only accounts for the mean shortfall (not higher moments).

This paper proposes a new risk-measure based on the mean-value under distorted probabilities. In addition to being coherent, this new risk-measure utilizes all the information contained in the loss distribution, and thus provides incentive for proactive risk management. By using distorted probabilities, this new risk-measure adequately accounts for extreme low-frequency and high-severity losses.

2. VaR as a Quantile Measure

Consider a risk portfolio (e.g., investment portfolio, trading book, insurance portfolio) in a specified time-period (e.g., 10-days, 1-year). Assume that the projected end-of-period aggregate loss (or shortfall) X has a probability distribution $F(x)$. With the prevalence of computer modeling based on scenarios and sampling, the distribution $F(x)$ is often discrete rather than continuous.

A standard risk-measure used by the banking industry is the Value-at-Risk, or VaR. It is an amount of money such that the portfolio loss will be less than that amount with a specified probability α (e.g., α=99%). More formally, we denote

$$\mathrm{VaR}(\alpha) = \mathrm{Min}\ \{x \mid F(x) \geq \alpha\}.$$

If the capital is set at VaR(α), the probability of ruin will be no greater than $1-\alpha$. For a discrete distribution, it is possible that $\Pr\{X > \mathrm{VaR}(\alpha)\} < 1-\alpha$.

Note that VaR is a risk-measure that only concerns about the frequency of default, but not the size of default. For instance, doubling the largest loss may not impact the VaR at all. Although being a useful risk-measure, VaR is short of being consistent when used for comparing risk portfolios.

3. Tail-VaR as a Coherent Risk-Measure

Artzner et al. (1999) advocated the following set of consistency rules for a coherent risk-measure:

1. Subadditivity: For all random losses X and Y, $\rho(X+Y) \leq \rho(X)+\rho(Y)$.
2. Monotonicity: If $X \leq Y$ for each outcome, then $\rho(X) \leq \rho(Y)$.
3. Positive Homogeneity: For positive constant b, $\rho(bX) = b\rho(X)$.
4. Translation Invariance: For constant c, $\rho(X+c) = \rho(X) + c$.

They demonstrated that VaR is not a coherent risk-measure. As an alternative, they advocated a risk-measure using Conditional Tail Expectation (CTE), which is also called Tail-VaR. Letting α be a prescribed security level, Tail-VaR has the following expression (see Hardy, 2001):

$$\mathrm{CTE}(\alpha) = \mathrm{VaR}(\alpha) + \frac{\Pr\{X > \mathrm{VaR}(\alpha)\}}{1-\alpha} \cdot \mathrm{E}\left[X - \mathrm{VaR}(\alpha) \mid X > \mathrm{VaR}(\alpha)\right].$$

This lengthy expression is due to the fact that for a discrete distribution we may have $\Pr\{X > \mathrm{VaR}(\alpha)\} < 1-\alpha$.

Tail-VaR reflects not only the frequency of shortfall, but also the expected value of shortfall. Tail-VaR is coherent, which makes it a superior risk-measure than VaR. The Office of the Superintendent of Financial Institutions in Canada has put in regulation for the use of CTE(0.95) to determine the capital requirement.

Recently there is a surge of interest in coherent risk-measures, evidenced in numerous discussions in academic journals and at professional conventions (see Yang and Siu, 2001; Meyers, 2001; among others).

The Tail-VaR, although being coherent, reflects only losses exceeding the quantile "VaR", and consequently lacks incentive for mitigating losses below the quantile "VaR". Moreover, Tail-VaR does not properly adjust for extreme low-frequency and high-severity losses, since it only accounts for the expected shortfall.

We argue that a good risk-measure should go beyond coherence. To this end, we introduce a family of coherent risk-measures based on probability distortions.

4. Distortion Risk-Measure

Definition 4.1. *Let g:[0,1]→[0,1] be an increasing function with g(0)=0 and g(1)=1. The transform F*(x)=g(F(x)) defines a distorted probability distribution, where "g" is called a distortion function.*

Note that F^* and F are equivalent probability measures if and only if g:[0,1]→[0,1] is continuous and one-to-one.

Definition 4.2. *We define a family of distortion risk-measures using the mean-value under the distorted probability F*(x)=g(F(x)):*

(4.1) $$r(\mathrm{X}) = \mathrm{E}^*(X) = -\int_{-\infty}^{0} g(F(x))dx + \int_{0}^{+\infty} [1-g(F(x))]dx\,.$$

The risk-measure ρ(X) in equation (4.1) is coherent when the distortion "g" is continuous (see Wang, Young, and Panjer, 1997).

The quantile-VaR corresponds to the distortion:

$$g(u) = \begin{cases} 0, & \text{when} \quad u < a\,, \\ 1, & \text{when} \quad u \geq a\,, \end{cases}$$

which shows a big-jump at u=α. This discontinuity pre-determines that VaR is not coherent.

The Tail-VaR corresponds to the distortion:

$$g(u) = \begin{cases} 0, & \text{when} \quad u < a\,, \\ \dfrac{u-a}{1-a}, & \text{when} \quad u \geq a\,, \end{cases}$$

which is continuous, but not differentiable at u=α. Note that "g" maps all percentiles below α to a single-point "0". Using this distortion "g" all information contained in that part of distribution will be lost.

Any smooth (differentiable) distortion "g" will give a coherent risk-measure that is different from Tail-VaR. Wirch and Hardy (1999) advocated using distortion risk-measure for capital requirement. They investigated a Beta family of distortion functions.

In this paper, we recommend the use of a special distortion known as the Wang Transform:

(4.2) $$g(u)=\Phi[\Phi^{-1}(u)-l],$$

where Φ is the standard normal cumulative distribution. The Wang Transform in equation (4.2) is a newly developed pricing formula that recovers CAPM and Black-Scholes formula under normal asset-return distributions (see Wang 2000). As shown in Wang (2001), equation (4.2) can also be derived from Buhlmann's (1980) equilibrium-pricing model. For a continuous distribution, the Wang Transform $F^*(X) = \Phi[\Phi^{-1}(F(x)) - l]$ is equivalent to an exponential titling $f^*(x) = c \cdot f(x) \cdot \exp(lz)$, with $z = \Phi^{-1}(F(x))$ being a standard normal percentile, and c being a re-scaling constant.

> **Definition 4.3.** *For a loss variable X with distribution F, we define a new risk-measure for capital requirement as follows:*
>
> 1. *For a pre-selected security level* α*, let* $\lambda = \Phi^{-1}(\alpha)$.
> 2. *Apply the Wang Transform*: $F^*(x) = \Phi[\Phi^{-1}(F(x)) - l]$.
> 3. *Set the capital requirement to be the expected value under* F^*:
>
> $$\mathrm{WT}(\alpha) = \mathrm{E}^*[X].$$

In this paper we shall refer to the risk-measure in Definition 4.3 as the *WT-measure*.

In Example 4.1 we compare the behaviors of the WT-measure with Tail-VaR.

Example 4.1. Consider two hypothetical portfolios with the following loss distributions.

Table 4.1. Portfolio A Loss Distribution

Loss x	Probability $f(x)$
\$0	0.600
\$1	0.375
\$5	0.025

Table 4.2. Portfolio B Loss Distribution

Loss x	Probability $f(x)$
\$0	0.600
\$1	0.390
\$11	0.010

Table 4.3. Risk-Measures With α=0.95.

Portfolio	CTE(0.95)	WT(0.95)
A	\$3.00	\$2.42
B	\$3.00	\$3.40

At the security level α=0.95, given that a shortfall occurs, Portfolios A and B have the same expected shortfall ($1.25). However, the maximal shortfall for Portfolio B ($11) is more than double that for portfolio A ($5). For most prudent individuals, Portfolio B constitutes a higher risk. Tail-VaR fails to recognize the differences between A and B. By contrast, the WT-measure gives a higher required capital for Portfolio B ($3.40) than for Portfolio A ($2.42).

It is desirable for a risk-measure to provide incentive for proactive risk-management. In Example 4.2 we illustrate that WT-measure encourages risk-management while Tail-VaR does not.

Example 4.2. Consider a risk portfolio with ten equally-likely scenarios with loss amounts $1, $2, …, $10, respectively. Assume that all loss-scenarios can be eliminated though active risk management, except that the worst-case $10 loss cannot be mitigated at all. Suppose a risk-manager is weighing the cost of risk-management against the benefit of capital relief. Tail-VaR would not encourage risk management, because there is no capital relief for removing losses below the worst-case loss. However, by removing all losses below $10, the WT-measure would always give a capital relief. For instance, using α=0.99, WT(α) drops from $9.71 to $8.52, showing a $1.19 capital relief; using α=0.95, WT(α) drops from $9.12 to $6.42, giving a $2.70 capital relief.

For a Normal(μ,σ^2) distribution, the Wang Transform gives another normal distribution with $\mu^*=\mu+\lambda\sigma$ and $\sigma^*=\sigma$. Therefore, for normal distributions, WT(α) is identical to VaR(α), the 100α-th percentile.

For distributions that are not normal, WT(α) may correspond to a percentile higher or lower than α, depending on the shape of the distribution, as shown in the following examples.

Example 4.3. When the loss X has a log-normal distribution with $\ln(X) \sim \text{Normal}(\mu,\sigma^2)$, the WT-measure has a simple formula:

$$WT(\alpha)= \exp(\mu+\lambda\sigma+\sigma^2/2) \text{ with } \lambda = \Phi^{-1}(\alpha).$$

The WT-measure for the log-normal distribution corresponds to the percentile $\Phi(\lambda+\sigma/2)$, which is higher than α.

Example 4.4. Consider an exponential distribution with mean=1. For α=0.99, we have WT(0.99)=5.02, VaR(0.99)=4.61, and CTE(0.99)=5.61. Note that WT(0.99) corresponds to the 99.34th percentile (higher than α).

Example 4.5. When the loss X has a Uniform[0.1] distribution, we have WT(0.99) = 0.95, which corresponds to the 95th percentile (lower than α).

For the WT-measure, risk diversification will result in lower λ for business units than for the whole company. This can be illustrated using a company consisting of two uncorrelated business units, each having a Normal(μ,σ^2) distribution. If the capital requirement is set at WT(0.99) at the company level, we have $\lambda = \Phi^{-1}(0.99)=2.326$ for the whole company. When the total capital is allocated equally to the two business units, we get $\lambda = \Phi^{-1}(0.99)/\text{Sqrt}(2) =1.645$ for each business unit. In other words, the required capital for each business unit is equal to WT(0.95).

5. Summary

We have shown that VaR, Tail-VaR, and the WT-measure are all members of the family of distortion risk-measures. Their differences are in the specific distortion "g":

- the "g" for VaR is neither continuous nor one-to-one.
- the "g" for Tail-VaR is continuous, but not differentiable or one-to-one.
- the "g" for the WT-measure is smooth and one-to-one.

The WT-measure is a direct application of the Wang Transform, which is an equilibrium-pricing transform that recovers CAPM and Black-Scholes formula. For normal distributions, the WT-measure corresponds to exactly the quantile-VaR. The WT-measure is not only coherent, but also reflects the whole loss distribution and thus provides incentive for risk-management.

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[1] Various other names have been used to describe this risk-measure, such as Tail Conditional Expectation (TCE) and Conditional Value-at-Risk (CVAR), etc.