A Universal Framework For
Pricing Financial And Insurance Risks

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Abstract

This paper presents a universal framework for pricing financial and insurance risks. Examples are given for pricing contingent payoffs, where the underlying asset or liability can be either traded or not traded. The paper also outlines an application of the framework to prescribe capital allocations within insurance companies, and to determine fair value for insurance liabilities.

INTRODUCTION

Currently there is a pressing need for a universal framework for the determination of the fair value of financial and insurance risks. In the insurance industry, this need is evident in the Society of Actuaries' "Symposium on Fair Value of Liabilities", and in the Casualty Actuarial Society's "Risk Premium Project" and "Task Force on Fair Valuing P/C Insurance Liabilities".

In the financial services industry, this pressing need is evidenced by the recent Basel Accords on regulatory risk management that require fair value, analogous to market prices, to be applied to all assets or liabilities, whether traded or not, on or off the balance sheet. In light of all these current events, this paper addresses a very timely subject.

The paper is comprised of three parts, summarized as follows:

Part One: The Framework introduces a new transform and correlation measure that extends CAPM to price all kinds of assets and liabilities, having any type of probability distribution, whether traded or underwritten, in finance or insurance. This transform is just as easily applied to contingent payoffs that are co-monotone with their underlying assets or liabilities.

In its simplest form, the new transform relies on a parameter called the "market price of risk", extending a familiar concept in CAPM to risks with non-normal distributions. The "market price of risk" can either be applied to, or implied from, a distribution, in order to arrive at a "risk-adjusted price" for the underlying risk in question. A "market price of risk" increases continuously with duration, and is consistent at each horizon date between an underlying and its co-monotone contingent payoff.

When returns for an underlying asset have a normal distribution, the new transform replicates the CAPM price for that underlying asset, and recovers the Black-Scholes price for options on that underlying asset.

Part Two: Examples of Pricing Contingent Payoffs illustrates applying the new framework to price call options on traded stocks, and to price weather derivatives.

Part Three: Capital Allocation and the Fair Value of Liabilities illustrates applying the new framework to insurance company capital allocations, and to the determination of fair value of insurance liabilities. In particular, it addresses a challenging issue concerning the long-term duration of liabilities. Also, the framework is equally applicable to primary insurance business and excess-of-loss reinsurance when calculating fair value of liabilities.

PART ONE. THE FRAMEWORK

Capital Asset Pricing Model

CAPM is a set of predictions concerning equilibrium expected returns on assets. Classic CAPM assumes that all investors have the same one-period horizon, and asset returns have multivariate normal distributions. For a fixed time horizon, let R_i and R_M be the rate-of-return for asset *i* and the market portfolio *M*, respectively. Classic CAPM asserts that

$$E[R_i] = r + \beta_i \{E[R_M] - r\},$$

where *r* is the risk-free rate-of-return and

$$\beta_i = \frac{Cov[R_i, R_M]}{\sigma_M^2}$$

is the *beta* of asset *i*.

Assuming that asset returns are normally distributed and the time horizon is one period (e.g., one year), a key concept in financial economics is the *market price of risk*:

$$\lambda_i = \frac{E[R_i] - r}{\sigma_i}.$$

In asset portfolio management, this is also called the *Sharpe Ratio*, after William Sharpe.

In terms of market price of risk, CAPM can be restated as follows:

$$\lambda_i = \frac{E[R_i] - r}{\sigma_i} = \frac{Cov[R_i, R_M]}{\sigma_i \sigma_M} \cdot \frac{E[R_M] - r}{\sigma_M} = \rho_{i,M} \cdot \lambda_M ,$$

where $\rho_{i,M}$ is the linear correlation coefficient between R_i and R_M. In other words, the market price of risk for asset *i* is directly proportional to the correlation coefficient between asset *i* and the market portfolio *M*.

CAPM provides powerful insight regarding the risk-return relationship, where only systematic risk deserves an extra risk premium in an efficient market. However, CAPM and the concept of “market price of risk” were developed under the assumption of multivariate normal distributions for asset returns. CAPM has serious limitations when applied to insurance pricing when loss distributions are not normally distributed. In the absence of an active market for insurance liabilities, the underwriting beta by line of business has been difficult to estimate.

Option Pricing Theory

Besides CAPM, another major financial pricing paradigm is modern option pricing theory, first developed by Fischer Black and Myron Scholes (1973).

Some actuarial researchers have noted that the payoff functions of a European call option and a stop-loss reinsurance contract are similar, and have proposed an “options pricing” approach to pricing insurance risks. Unfortunately, the Black-Scholes formula only applies to lognormal distributions of market returns, whereas actuaries work with a large array of distributional forms.

Furthermore, there are subtle differences between option pricing and actuarial pricing (see Mildenhall, 1999). One way to better appreciate the difference between "financial asset pricing" and "insurance pricing," is to recognize the difference in types of data available for pricing.

Options pricing is performed in a world of *Q*-measure, where the available data consists of observed market prices for related financial assets. On the other hand, actuarial pricing is conducted in a world of *P*-measure, where the available data consists of projected losses, whose amounts and likelihood need to be converted to a "fair value" price (see Panjer et al, 1998).

Because of this difference, modern option pricing is mostly concerned with the minimal cost of setting up a hedging portfolio, whereas actuarial pricing is based on actuarial present value of costs, with additional adjustments for correlation risk, parameter uncertainty and cost of capital.

A Universal Pricing Method

Consider a financial asset or liability over a time horizon $[0,T]$. Let $X=X_T$ denote its future value at time $t=T$, with a cumulative distribution function $F(x)=\Pr\{X\le x\}$. In Wang (2000), the author proposed a universal pricing method based on the following transform:

$$F^*(x) = \Phi\left[\Phi^{-1}(F(x)) + \lambda\right], \qquad (1)$$

where Φ is the standard normal cumulative distribution. The key parameter λ is called the *market price of risk*, reflecting the level of systematic risk. The transform (1) is now better known as the *Wang Transform* among financial engineers and risk managers. The Wang Transform was partly inspired by the work of several prominent actuarial researchers, including Gary Venter (1991, 1998) and Robert Butsic (1999).

For a given asset X with $F(x)$, the Wang Transform will produce a "risk-adjusted" cumulative probability distribution $F^*(x)$. The mean value under $F^*(x)$, denoted by $E^*[X]$, will define a risk-adjusted "fair value" of X at time T, which can be further discounted to time zero, using the risk-free interest rate.

The Wang Transform is fairly easy to numerically compute. Many software packages have both Φ and Φ^{-1} as built-in functions. In Microsoft Excel, $\Phi(y)$ can be evaluated by NORMDIST(y,0,1,1) and $\Phi^{-1}(y)$ can be evaluated by NORMINV(y,0,1).

One fortunate property of the Wang Transform is that normal and lognormal distributions are preserved:

- If F has a Normal(μ,σ^2) distribution, F^* is also a normal distribution with $\mu^* = \mu-\lambda\sigma$ and $\sigma^* = \sigma$.
- If F has a lognormal(μ,σ^2) distribution such that $\ln(X) \sim \text{Normal}(\mu,\sigma^2)$, F^* is

another lognormal distribution with $\mu^* = \mu - \lambda\sigma$ and $\sigma^* = \sigma$.

Stock prices are often modeled by lognormal distributions, which implies that stock returns are modeled by normal distributions. Equivalent results can be obtained by applying the Wang Transform either to the stock price distribution, or, to the stock return distribution.

Consider an asset *i* on a one-period time horizon. Assume that the return R_i for asset *i* has a normal distribution with a standard deviation of σ_i. Applying the Wang Transform to the distribution of R_i we get a risk-adjusted rate-of-return:

$$E^*[R_i] = E[R_i] - \lambda\sigma_i \, .$$

In a competitive market, the risk-adjusted return for all assets should be equal to the risk-free rate, *r*. Therefore we can infer that $\lambda = (\mathrm{E}[R_i] - r)/\ \sigma_i$, which is exactly the same as the market price of risk in classic CAPM. With λ being the market price of risk for an asset, the Wang Transform replicates the classic CAPM.

Unified Treatment of Assets & Liabilities

A liability *X* can be viewed as a negative asset $Y = -X$, and vice versa. Mathematically, if a liability has a market price of risk λ, when treated as a negative asset, the market price of risk will be $-\lambda$. That is, the market price of risk will have the same value but opposite signs, depending upon whether a risk vehicle is treated as an asset or liability. For a liability *X*, the Wang Transform has an equivalent representation.

$$S^*(x) = \Phi\left[\Phi^{-1}(S(x)) + \lambda\right], \qquad (2)$$

where $S(x) = 1 - F(x)$.

If a liability has a Normal(μ,σ^2) distribution, the Wang Transform will produce another normal distribution with $\mu^* = \mu + \lambda\sigma$ and $\sigma^* = \sigma$. Thus, for a liability with a normal distribution, the Wang Transform recovers the traditional standard-deviation loading principle, with the parameter λ being the constant multiplier.

A New Measures of Correlation

According to CAPM, the market price of risk λ should reflect the correlation of an asset with the overall market portfolio. When we generalize the concept of market price of risk to assets and liabilities with non-normal distributions, the Pearson linear correlation coefficient becomes an inadequate measure of correlation. Examples can be constructed such that a deterministic relationship has a Pearson correlation coefficient close to zero. Such an example was provided in Wang (1998):

> Consider the case where $X\sim$ lognormal(0,1) and $Y = X^\sigma$. Despite this deterministic relationship, the linear correlation coefficient between *X* and *Y* approaches zero as σ increases to infinity. That is, $\rho_{X,Y} \rightarrow 0$ as σ increases.

This also implies that correlation should not be estimated by running linear regression, unless all of the variables have normal distributions.

Now we show a new way to extend the Pearson correlation coefficient to variables with non-normal distributions. For any pair of variables $\{X, Y\}$ with distributions F_X and F_Y, we transform them into "standard normal variables":

$$U=\Phi^{-1}[F_X(X)], \quad \text{and} \quad V=\Phi^{-1}[F_Y(Y)].$$

We next define a new measure of correlation between $\{X, Y\}$ as the Pearson linear correlation coefficient between these transformed "standard normal variables" $\{U, V\}$:

$$\rho^*_{X,Y} = \frac{Cov(U,V)}{\sigma(U)\cdot\sigma(V)} = Cov(U,V)\,.$$

Now, let us reconsider the case where $X\sim$ lognormal(0,1) and $Y=X^{\sigma}$. Consistent with this deterministic relationship, this new measure of correlation between X and Y is always 1. That is, $\rho^*_{X,Y}=1$ for all σ values.

Using our new measure of correlation we may extend classic CAPM as follows:

$$\lambda_i = \rho^*_{i,M}\cdot\lambda_M\,,$$

where λ_i and λ_M are the respective market prices of risk in the Wang Transform, without assuming normality.

Pricing of Contingent Payoffs

For an underlying risk X and a function h, we say that $Y = h(X)$ is a derivative (or contingent payoff) of X, since the payoff of Y is a function of the outcome of X. If the function h is monotone, we say that Y is a co-monotone derivative of X. For example, a European call option is a co-monotone derivative of the underlying asset; in (re)insurance, an excess layer is a co-monotone derivative of the ground-up risk.

Theoretically, the underlying risk X and its co-monotone derivative Y should have the same level of systematic risk, λ, simply because that they have the same correlation (as shown by using our new measure of correlation) with the market portfolio.

In pricing a contingent payoff $Y = h(X)$, there are two ways of applying the Wang Transform.

- Method I: Apply the Wang Transform to the distribution F_X of the underlying risk X. Then derive a risk-adjusted distribution F^*_Y from F^*_X using $Y^* = h(X^*)$.
- Method II: First derive its own distribution F_Y for $Y = h(X)$. Then apply the Wang Transform to F_Y directly, using the same λ as in Method I.

Mathematically it can be shown that these two methods are equivalent. This important result validates using the Wang Transform for risk-neutral valuations of contingent payoffs.

Implied λ and the Effect of Duration

For a traded asset, the market price of risk λ can be estimated from observed market data. We shall now take a closer look at the implied market price of risk and how it varies with the time horizon under consideration.

Consider a continuous time model where asset prices are assumed to follow a geometric Brownian motion (GBM). Consider an individual stock, or a stock index, *i*. The asset price $X_i(t)$ satisfies the following stochastic differential equation:

$$\frac{dX_i(t)}{X_i(t)} = \mu_i dt + \sigma_i dW_i\,, \qquad (4)$$

where dW_i is a random variable drawn from a normal distribution with mean equal to zero and variance equal to dt. In equation (4), μ_i is the expected rate of return for the asset, and σ_i is the volatility of the asset return. Let $X_i(0)$ be the current asset price at time zero. For any future time T, the prospective stock price $X_i(T)$ as defined in equation (4) has a lognormal distribution (see Hull, 1997, p. 229):

$$X_i(T)/X_i(0) \;\sim\; \text{lognormal}\left(\mu_i T - 0.5\sigma_i\sqrt{T},\ \sigma_i^2 T\right). \qquad (5)$$

Next we apply the Wang Transform to the distribution of $X_i(T)$ in (5) and we get

$$X_i^*(T)/X_i(0) \;\sim\; \text{lognormal}\left(\mu_i T - \lambda\sigma_i\sqrt{T} - 0.5\sigma_i\sqrt{T},\ \sigma_i^2 T\right).$$

For any fixed future time T, a "no arbitrage" condition (or simply, the market value concept) implies that the risk-adjusted future asset price, when discounted by the risk-free rate, must equal the current market price. In this continuous-time model, the risk-free rate r needs to be compounded continuously.

As a result, we have an implied parameter value:

$$\lambda = \lambda_i(T) = \frac{(\mu_i - r)}{\sigma_i}\sqrt{T} = \sqrt{T}\cdot\lambda_i(1)\,. \qquad (6)$$

The implied λ in (6) coincides with the market price of risk of asset *i* as defined in Hull (1997, p. 290). This implied λ is also consistent with Robert Merton's inter-temporal, continuous-time CAPM (see Merton, 1973).

It is interesting to note that the market price of risk λ increases as the time horizon lengthens. This makes intuitive sense since the longer the time horizon, the greater the exposure to unforeseen changes in the overall market environment. This interesting result has applications in pricing long-tailed insurance where losses are not reported or settled until many years after the policy period expires.

If the evolution of incurred loss resembles geometric Brownian motion, the parameter λ should be proportional to the square root of the time period from policy inception to the date of loss settlement. The relationship (6) between λ and duration T is very useful in calculating fair value of insurance liabilities (including loss reserve discounting) and optimizing capital allocations within an insurance company.

Applying the Wang Transform with the λ in equation (6), asset i has a risk-adjusted distribution

$$X_i^*(T)/X_i(0) \;\sim\; \text{logormal}\left(rT - 0.5\sigma_i\sqrt{T}, \sigma_i^2 T\right),$$

where both the market price of risk $\lambda_{i,}$ and the expected stock return μ_i have dropped out from the transformed distribution $F^*(x)$.

Recovery of the Black-Scholes Formula

A European call option on an underlying stock (or stock index) i with a strike price K and exercise date T is defined by the following payoff function

$$Y = \text{Call}(K) = \begin{cases} 0, & \text{when } X_i(T) \le K, \\ X_i(T) - K, & \text{when } X_i(T) > K. \end{cases}$$

Being a non-decreasing function of the underlying stock price, the option payoff, Call(K), is co-monotone with the terminal stock price, $X_i(T)$; thus it has the same level of systematic risk as the underlying stock i. Therefore, the same λ as in equation (6) should be used to price the option Call(K). In other words, the price of a European call option is the expected payoff under the transformed (risk-neutral) stock price distribution $F^*(x)$, where the expected stock return μ_i is replaced by the risk-free rate r. The resulting option price is exactly the same as the Black-Scholes formula.

There is an analogy between an unlimited stop-loss cover with retention K, and a European call-option with strike price K. Both are co-monotone derivatives of the underlying (loss or asset) variable. By applying the Wang Transform to the stop-loss variable, we get a stop-loss premium as the expected stop-loss value under the transformed ground-up loss distribution.

Likewise, the price for a European call option can be evaluated as the expected option payoff under the transformed (risk-neutral) distribution for the underlying stock price, where the expected stock-return μ_i does not appear in the options pricing model. Thus the Wang Transform adds a new perspective to the well-known risk-neutral valuation methodology of options (see Cox and Ross, 1976).

Equilibrium and Replication Perspectives

Recall that CAPM provides an equilibrium perspective of asset prices in light of its correlation with the market portfolio. With the equilibrium perspective, an option is co-monotone with the underlying asset, thus have the same market price of risk. Using the same market price of risk, the Wang Transform produces an option price as the expected option payoff under a transformed "risk-neutral" asset distribution where the expected rate-of-return is equal to the risk-free rate-of-return.

On the other hand, modern finance presents the Black-Scholes formula via a replication perspective. The replication approach relies on the ability to create a continuous riskless hedge. If asset prices change in small amounts, it is possible to simultaneously buy an option and sell a quantity of the underlying asset, so that the combined portfolio has no risk. Note that the instantaneous hedge is possible only because the option is a co-monotone derivative of the underlying asset.

Emanuel Derman (1996), who had worked closely with Fischer Black, commented that "Deep inside, Fischer seemed to rely on the equilibrium approach of the capital asset pricing model as the source for his intuition about options pricing. I believe this is the way the Black-Scholes equation was originally derived, although the first derivation of the options pricing formula in the Black-Scholes article is based on valuation by replication."

The Wang Transform takes the equilibrium perspective of CAPM, and yet is able to reproduce the Black-Scholes price for options on underlying assets with lognormal distributions. The Wang Transform thus formalizes an intrinsic relationship between CAPM and the Black-Scholes formula, along the lines of Fischer Black's reported insights.

Adjust for Parameter Uncertainty

Before applying the Wang Transform, some adjustments for parameter uncertainty must be incorporated. This can be done explicitly on a case-by-case basis using Bayesian-type prior mixing parameters. One non-Bayesian method of adjusting for parameter uncertainty uses a positive parameter b, and modifies the objective probabilities as follows:

$$F^*(x) = \Phi\left[b \cdot \Phi^{-1}(F(x))\right]. \qquad (7)$$

For adjustment (7), the mean value of the distribution may not be preserved for non-symmetric distributions. The composite of transforms (7) and (1) incorporates both systematic risk and parameter uncertainty, and produces a more generalized version of the Wang Transform:

$$F^*(x) = \Phi\left[b \cdot \Phi^{-1}(F(x)) + \lambda\right], \qquad (8)$$

When a marketplace is rational and ambiguity-averse, the value of *b* should be greater than 1 for assets, and less than 1 for liabilities. The adjustment factor *b* may be tweaked for the value of $F(x)$, with further adjustments possible for extremes, like for way out-of-the-money contingent claims, or way-beyond-a-horizon-date claim settlements, where markets are illiquid, benchmark data sparse, negotiations difficult, and the cost of keeping capital reserves is high.

For a lognormal distribution, the transform (8) amplifies the volatility parameter after a location parameter shift, along lines suggested by Butsic (1999). Gary Venter, in a private communication, has also reported to the author that John Major had fitted transform (8) to empirically observed property CAT treaty prices.

Extrapolation of Tail Probabilities

Using transform (8) to adjust for parameter uncertainty does not always work in all situations. For instance, an insurance contract might offer a $100M limit, with no data indicating historical losses greater than $50M, even after trending.

In such a case, tail probabilities for losses greater than 50M need to be extrapolated from the estimated probabilities for losses below 50M. The Extreme Value Theory may be a useful technique for the extrapolation (see Embrechts, et al, 1997). The Wang Transform can be applied to the extrapolated tail probabilities.

Pricing Versus Portfolio Management

Pricing is a result of collective market behaviors. The market price of risk reflects not only the correlation with the overall market portfolio, but also an average cost of capital commitment for a given industry.

Portfolio management involves an active selection (deletion) of the most (least) profitable business in relation to the incremental risk to the existing portfolio. Fortunately, the Wang Transform is as useful in portfolio management as it is in pricing. For example, the Wang Transform is more precise than a standard deviation as an underlying axis for plotting an efficient frontier, or for calculating an optimal portfolio. The Wang Transform can also be used by a portfolio manager to identify good/bad risks by comparing their respectively implied lambdas with their own benchmarks for risk and return.

PART TWO. EXAMPLES OF PRICING CONTINGENT PAYOFFS

A contingent payoff is a contractual agreement between counter-parties, whose payment trigger and amount are determined by observed outcomes of the underlying variable. A contingent payoff is a more general type of financial instrument than an option, since the underlying variable can include non-traded assets or liabilities, statistical indices, or even physical events. Most underlying variables do not follow a lognormal distribution, making the Black-Scholes formula inappropriate for benchmark pricing. In contrast, the Wang Transform is applicable to any distributional form, and can be used as a universal method for pricing all kinds of contingent payoffs.

An Example of Pricing Options

Asset pricing is based on anticipated future price movements. Historical returns may or may not be a good indicator of future price movement. For illustration purposes, we assume the availability of a robust stock price projection model utilizing historical price data and other available information. Such a stock price projection may be based on a GARCH model with due considerations to mean-reversion and other economic factors. For our illustration, such a model has produced the below sample of outcomes with equal probability weights.

The underlying is a stock index with a current price of $1326.03. Our model has produced 20 outcomes (partially based on 5-year history of quarterly returns):

> 1218.71, 1309.51, 1287.08, 1352.47, 1518.84, 1239.06, 1415.00, 1387.64, 1602.70, 1189.37, 1364.62, 1505.44, 1358.41, 1419.09, 1550.21, 1355.32, 1429.04, 1359.02, 1377.62, 1363.84.

The stock index return has a mean of 4.08% and a standard deviation of 8.07%. Assuming that the 3-month risk-free rate is 1.5%. The empirical “Sharpe Ratio” for the 3-month time-horizon is 0.32=(4.08%–1.5%)/8.07%.

We want to price a 3-month European call option on this stock with a strike price of $1375. We will apply the Wang Transform to the sample stock index distribution, *without assuming a lognormal distribution*. Detailed steps are shown in Table 1 with further explanations below:

- Column 1. Sort the sample of projected outcomes in ascending order, from worst to best. In this example, the underlying is considered to be an asset, so the worst outcome here is the lowest number, and best is the highest number.
- Column 2. Assign objective probabilities f(x) to each projected outcome x. In this example, the objective probability is based on 20 equally weighted observations, so each assigned probability is 1/20.
- Column 3. Add up the individual objective probabilities f(x) to yield a series of cumulative probabilities F(x).
- Column 4. Using the empirical Sharpe Ratio (0.32) as a “starter” lambda value, apply

the Wang Transform to the cumulative probabilities F(x), to yield F*(x). Recall that the Sharpe Ratio assumes a normal distribution, so we may need to tweak our lambda value later, to account for a possible non-normal distribution.

- Column 5. De-cumulate the transformed probabilities F*(x) to recover f*(x). Evaluate the mean value of this projected sample using probability weights f*(x). If the discounted mean value is greater (or less) than the current market value, tweak upward (or downward) the lambda value. Repeat the process of columns 4-5 until the discounted mean value matches the current market price. In this example, the “starter” lambda value of 0.320 has been tweaked to 0.342, in order to match the current price of $1326.03. The values of F*(x) and f*(x) shown in columns 4 and 5 are thus the final transformed probabilities using λ=0.342. Now we proceed to columns 6-8.
- Column 6. For a given strike price ($1375 in this example), calculate the option payoff for each projected future price for the stock. That is, y(x)=max(x–1375, 0).
- Column 7. Calculate the expected payoff by multiplying the values of the option payoff function in Column 6 by the objective probabilities in Column 2. In this example, the resulting expected payoff is $41.53 before discounting, and $40.93 after discounting.
- Column 8. Calculate the risk-adjusted payoff using the transformed distribution. We do that by multiplying Column (5) by Column (6). The resulting option price is $25.35 before discounting, and $24.98 after discounting.

Table 1. Pricing of Call-Option Using the Wang Transform (λ=0.342)

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Sorted	**Objective**		**Transformed**		**Contingent**	**Weighted**	**Risk**
Sample	**Probability**		**Probability**		**Payoff**	**Value**	**Adjusted**
x	**f(x)**	**F(x)**	**F*(x)**	**f*(x)**	**y(x)**	**f(x) y(x)**	**f*(x) y(x)**
1,189.37	0.05000	0.05000	0.0963	0.0963	-	-	-
1,218.71	0.05000	0.10000	0.1737	0.0774	-	-	-
1,239.06	0.05000	0.15000	0.2437	0.0700	-	-	-
1,287.08	0.05000	0.20000	0.3087	0.0650	-	-	-
1,309.51	0.05000	0.25000	0.3698	0.0611	-	-	-
1,352.47	0.05000	0.30000	0.4276	0.0579	-	-	-
1,355.32	0.05000	0.35000	0.4827	0.0551	-	-	-
1,358.41	0.05000	0.40000	0.5353	0.0526	-	-	-
1,359.02	0.05000	0.45000	0.5856	0.0503	-	-	-
1,363.84	0.05000	0.50000	0.6338	0.0482	-	-	-
1,364.62	0.05000	0.55000	0.6800	0.0462	-	-	-
1,377.62	0.05000	0.60000	0.7242	0.0442	2.62	0.13	0.12
1,387.64	0.05000	0.65000	0.7665	0.0423	12.64	0.63	0.53
1,415.00	0.05000	0.70000	0.8069	0.0404	40.00	2.00	1.62
1,419.09	0.05000	0.75000	0.8453	0.0384	44.09	2.20	1.69
1,429.04	0.05000	0.80000	0.8817	0.0364	54.04	2.70	1.97
1,505.44	0.05000	0.85000	0.9160	0.0342	130.44	6.52	4.47
1,518.84	0.05000	0.90000	0.9478	0.0318	143.84	7.19	4.57
1,550.21	0.05000	0.95000	0.9765	0.0288	175.21	8.76	5.04
1,602.70	0.05000	1.00000	1.0000	0.0235	227.70	11.38	5.34
Values							
Expected	1,380.15			1,346.07		41.53	25.35
Discounted	**1,359.60**			**1,326.03**		**40.91**	**24.98**

Here are some related comments:

- The market price of risk, as calculated by (E[R]-r)/σ, is precise only when the underlying asset has a normal distribution. The Wang Transform, on the other hand, can iterate a precise market price of risk for underlying assets or liabilities with any type of distribution.
- The Wang Transform can be used to alleviate some of the volatility smiles in observed option prices. This is because the Wang Transform can automatically incorporate any deviations (like skewness and kurtosis) from normality in the asset return distribution.
- With the Wang Transform, we can take advantage of a good price projection model incorporating stochastic volatilities for underlying asset.

An Example of Pricing Weather Derivatives

For most weather derivatives, a payoff is contingent upon the number of observed Heating-Degree-Days (HDD) for the winter months, or Cooling-Degree-Days (CDD) for the summer months, multiplied by some notional amount. The underlying variables of weather derivatives, namely HDDs and CDDs, are not traded assets by themselves. This is in contrast to equity derivatives, where the underlying stock is usually a traded asset. Therefore, to price weather derivatives, an equilibrium approach makes more sense than a replication approach.

In winter months, extreme cold weather drives up the cost for heating. The recent energy crisis has boosted the demand for call options on HDDs, to hedge rising heating costs. The writers of such options need to set aside capital to fund potential payouts.

We give an example of using the Wang Transform to price weather derivatives. Here we use Chicago Mercantile Exchange Weather Data --- Monthly Aggregate from 1/1/1979 to 1/1/2001.

Table 2 is the aggregate HDDs for months of December observed at the Chicago O'Hare Station. Note that there are a total of 22 observations with a mean of 1154.7 and a standard deviation of 193.4.

Table 2. Monthly Aggregate Data for Chicago O'Hare Station, 1979-2000

Date	Dec-79	Dec-80	Dec-81	Dec-82	Dec-83	Dec-84	Dec-85	Dec-86
HDD	972.5	1147.0	1244.0	901.0	1573.0	1055.0	1488.0	1065.5
Date	Dec-87	Dec-88	Dec-89	Dec-90	Dec-91	Dec-92	Dec-93	Dec-94
HDD	1018.5	1155.0	1474.5	1129.5	1077.5	1129.5	1090.5	938.5
Date	Dec-95	Dec-96	Dec-97	Dec-98	Dec-99	Dec-00		
HDD	1199.5	1156.0	1040.0	940.5	1090.5	1517.5		

Table 3. Pricing Weather Derivatives Using the Wang Transform (λ=–0.25)

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Sorted	**Objective**		**Transformed**		**Contingent**	**Weighted**	**Risk**
Sample	**Probability**		**Probability**		**Payoff**	**Value**	**Adjusted**
x	**f(x)**	**F(x)**	**F*(x)**	**f*(x)**	**y(x)**	**y(x) f(x)**	**y(x) f*(x)**
901.0	0.0455	0.0455	0.0262	0.0262	0.0	0.0	0.0
938.5	0.0455	0.0909	0.0565	0.0303	0.0	0.0	0.0
940.5	0.0455	0.1364	0.0890	0.0326	0.0	0.0	0.0
972.5	0.0455	0.1818	0.1233	0.0343	0.0	0.0	0.0
1018.5	0.0455	0.2273	0.1592	0.0358	0.0	0.0	0.0
1040.0	0.0455	0.2727	0.1964	0.0372	0.0	0.0	0.0
1055.0	0.0455	0.3182	0.2349	0.0385	0.0	0.0	0.0
1065.5	0.0455	0.3636	0.2747	0.0398	0.0	0.0	0.0
1077.5	0.0455	0.4091	0.3157	0.0410	0.0	0.0	0.0
1090.5	0.0455	0.4545	0.3579	0.0422	0.0	0.0	0.0
1090.5	0.0455	0.5000	0.4013	0.0434	0.0	0.0	0.0
1129.5	0.0455	0.5455	0.4460	0.0447	0.0	0.0	0.0
1129.5	0.0455	0.5909	0.4920	0.0460	0.0	0.0	0.0
1147.0	0.0455	0.6364	0.5393	0.0474	0.0	0.0	0.0
1155.0	0.0455	0.6818	0.5882	0.0488	0.0	0.0	0.0
1156.0	0.0455	0.7273	0.6385	0.0504	0.0	0.0	0.0
1199.5	0.0455	0.7727	0.6907	0.0522	0.0	0.0	0.0
1244.0	0.0455	0.8182	0.7449	0.0542	0.0	0.0	0.0
1474.5	0.0455	0.8636	0.8014	0.0566	124.5	5.7	7.0
1488.0	0.0455	0.9091	0.8611	0.0596	138.0	6.3	8.2
1517.5	0.0455	0.9545	0.9252	0.0641	167.5	7.6	10.7
1573.0	0.0455	1.0000	1.0000	0.0748	223.0	10.1	16.7
					Total	*29.7*	*42.7*
						Exp.Value	*Price*

For illustration, the call option on Dec-2001-HDDs is assumed to have a strike price of 1350. Thus, the payoff function can be expressed mathematically as max(HDD–1350, 0). In order to apply the Wang Transform, we first sort the annual December HDDs in an ascending order and assign objective probabilities. Here we use historical data without adjusting for on-going trends or cycles of weather conditions. In real life applications, of course, such trends and cycles can be considered.

The key to the application of the Wang Transform boils down to the selection of the lambda value. This lambda value should be decided from observed market prices wherever possible. In this case, the lambda value is not known, and the table of outcomes only reflects HDDs as a denominational unit of value. No face value or notional amount has been attached to HDDs as yet.

Table 3 shows the application of the Wang Transform. For the strike level of 1350 Dec-2001-HDDs, the call option has an expected payoff of 29.68 Dec-2001-HDDs, using objective probabilities. However, the "fair value" of the option is 42.70 Dec-2001-HDDs, using the transformed distribution. We only need to apply a common notional amount, denominated in a common currency, and a risk-free rate for that currency, to arrive at a discounted cash price for this option.

Applying a face value of $1 for each HDD, we can apply the same lambda of –0.25, so that call options with different strike prices can be evaluated and compared (see Table 4).

Table 4. Price of Options at Various Strike Prices (λ=–0.25)

Strike	1250	1300	1350	1400	1450	1500
Exp. Payoff	$ 47.86	$ 38.77	$ 29.68	$ 20.59	$ 11.50	$ 4.11
Price	$ 68.21	$ 55.45	$ 42.70	$ 29.94	$ 17.18	$ 6.59
Loading	43%	43%	44%	45%	49%	60%

Historical data may not guarantee an accurate indication of future weather conditions, given the rapid environmental changes. There are many ways of adjusting for parameter uncertainty. Here we consider one adjustment using transform (8) with a constant multiplier b=0.95. To somewhat offset the extra loading, we adjusted the lambda value downward to –0.20. As shown in Table 5, with adjustment for parameter uncertainty, the relative loading increases faster with strike prices.

Table 5. Price of Options with adjustment for parameter uncertainty (λ=–0.20, b=0.95)

Strike	1250	1300	1350	1400	1450	1500
Exp. Payoff	$ 47.86	$ 38.77	$ 29.68	$ 20.59	$ 11.50	$ 4.11
Price	$ 68.28	$ 55.60	$ 42.92	$ 30.24	$ 17.55	$ 6.93
Loading	43%	43%	45%	47%	53%	69%

Comments:

- The lognormal assumption is absolutely not needed here.
- The methodology is analogous to the Black-Scholes valuation for options.

PART THREE. CAPITAL ALLOCATION & FAIR VALUE OF LIABILITIES

Next we discuss the application of the Wang Transform to insurance capital allocation and to the calculation of the fair value of liabilities. Consider an insurance company writing multiple lines of business. Assume that we already know the overall economic capital for the company, or alternatively, we have derived a total required economic capital for the company based on industry benchmarks. Our goal is to allocate the cost of capital to different lines-of-business and individual contracts. Given the long-tailed nature of insurance payment patterns, insurers are required to continuously hold capital to support the reserve liabilities. The underlying issue is how to appropriately reflect the duration of insurance liabilities.

Some actuaries suggest that capital needs to be committed each year in proportion to all remaining unpaid losses, without consideration of the diversification effect among development years. Others argue that the diversification benefit between development years should be considered. Some even go to the extreme that only a one-time allocation is needed in the first year to account for the uncertainties associated with the present value of reserves. Dramatically different implications of these viewpoints present a challenging issue associated with insurance capital allocation. The rest of the paper is devoted to tackling this issue using the pricing framework in Part One.

Available Data

We first consider *ground-up* or *primary* business only. We shall use the following data:

- Based on historical *accident-year* ultimate loss ratios, we have estimates of the loss ratio volatility for each lines of business, denoted by σ_{AY}.
- We have estimates of the loss payment pattern for each line of business, with an average duration, denoted by D_{GU}. Let $R(t)$ be the portion of losses remaining unpaid by time t. We have $D_{GU} = \int_0^\infty R(t)dt$.

Assumptions for the Evolution of Losses

1. The best-estimate of remaining unpaid losses evolves with the passage of time as more information becomes available. During each time period (e.g. one-year), the revised estimates of loss reserve may go up or go down, with a random nature.
2. There are two opposing arguments regarding the relative uncertainty of the remaining reserve: (a) it should increase with time as more risky claims are settled later; (b) it should decrease with time as more information becomes available. Here we assume that the relative uncertainty (coefficient of variation) of remaining reserves remains constant over time. See Philbrick (1994) for further discussion of this issue.
3. Based on the above considerations, we assume a geometric Brownian motion process for the loss reserve evolution over time. Assuming that the instantaneous per annum volatility is a constant σ_1, we have

$$\sigma_{AY}^2 = \int_0^\infty \sigma_1^2 \cdot R(t)dt = \sigma_1^2 \cdot D_{GU} \,.$$

Thus we can estimate the per-year volatility as

$$\sigma_1 = \frac{\sigma_{AY}}{\sqrt{D_{GU}}} \,.$$

In practice, the geometric Brownian motion assumption can be further refined to reflect more closely the true underlying process. The instantaneous volatility σ(t) may change with time. For property lines of business, σ(t) may be higher for the pricing risk (0<t<1) than the reserving risk (t>1), as new information will emerge during the contract period regarding catastrophe activities. For casualty lines, σ(t) may be higher for IBNR reserves than that for case reserves. In the case of changing σ(t), an average per annum volatility can be calculated by

$$\sigma_1 = \frac{\int_0^\infty \sigma(t)R(t)dt}{\int_0^\infty R(t)dt} \,.$$

Although the mentioned refinements can be incorporated in the calibration of the insurance company capital allocation, here we will restrict ourselves to the geometric Brownian motion assumption with $\sigma(t)= \sigma_1$.

Measure of Risk

We define a risk-measure to approximate the *cost* of capital commitment, based on the following assumptions:

(a) The cost of capital per-year is proportional to the underlying *per-year* volatility σ_1.

(b) According to the multi-period CAPM, the magnitude of systematic risk increases with the time horizon. Let parameter λ_1 be the per-year "systematic risk". The multi-period CAPM says that the systematic risk for time horizon *T* is: $\lambda_T = \lambda_1 \cdot \sqrt{T}$. Intuitively this makes sense. For liability insurance, the longer the duration, the higher the uncertainty, especially with respect to judicial changes and court rulings.

(c) The cost of capital is proportional to the "systematic risk" for the underlying business. Financial economists may think of systematic risk as the "correlation risk" relative to the market portfolio. Under financial theory, insurance insolvency can be automatically reflected in the market insurance prices, the required capital would be much less than that by rating agencies and regulators. However, in the presence of transaction costs, the systematic risk should reflect not only the correlation risk, but also the true cost of doing business including the cost of capital. *As a result, we assume that most lines of business have a similar level of systematic risk, namely, λ_1, except for property catastrophe and mass-torts in which higher λ values should be used.*

For each $1 of expected loss for a line of business with per-annum volatility σ_1 and average duration of D_{GU}, the total risk measure for ground-up insurance coverage is:

$$\lambda_1 \cdot \sigma_1 \cdot D_{GU} = \lambda_1 \cdot \frac{\sigma_{AY}}{\sqrt{D_{GU}}} \cdot D_{GU} = \lambda_1 \cdot \sigma_{AY} \cdot \sqrt{D_{GU}} = \lambda_{GU} \cdot \sigma_{AY},$$

where $\lambda_{GU} = \lambda_1 \cdot \sqrt{D_{GU}}$. This relation can also be verified using a steady-state model.

Pricing Ground-up Insurance Contracts

For a given line of business, to calculate the insurance premium for each $1 ground-up expected loss, we will do the following:

(1). Calculate the discount factor $\mathrm{PV}_{GU}(1)$: Use market risk-free interest rate and the *ground-up* loss payment pattern.

(2). Apply risk loading to derive a pure premium:

$$PV_{GU}(1) \cdot \{1 + \lambda_1 \sigma_1 D_{GU}\}.$$

The factor λ_1 should be calibrated from "total portfolio re-balancing" based on a target return-on-equity (TROE). In other words, for the aggregate insurance portfolio, the ratio of "the total risk load plus investment return" to the total economic capital should produce a target return-on-equity. The total allocated capital over the lifetime of this $1 liability is

$$\lambda_1 \sigma_1 D_{GU}(1+r)/(TROE - r),$$

For year j, the allocated capital is

$$\lambda_1 \sigma_1 P_j (1+r)/(TROE - r),$$

where P_j is the expected percentage of losses to be paid within year *j*.

(3). Load for expenses: suppose the total expense factor is θ, we can load the pure premium by a factor of $1/(1-\theta)$.

(4) Knowing the amount of allocated capital, we can calculate the actual ROE for any given quoted premium rate.

A remark on the relation to the Wang Transform: Assume that ground-up accident-year loss ratio follows a Brownian motion process with a total volatility $\sigma_{AY.}$ Formula (4) is an approximation to the resulting premium using the Wang Transform with $\lambda_{GU} = \lambda_1 \cdot \sqrt{D_{GU}}$. Thus, for ground-up business, our risk load (and capital allocation) methodology is shown to be an approximate result of the Wang Transform.

Pricing Aggregate Stop-Loss Insurance Contracts

The Wang Transform can be applied to pricing aggregate stop-loss layers, as a direct analogy to the Black-Scholes formula for pricing options. Aggregate covers are usually applied on a multi-line basis. One needs to blend the volatility and payment patterns by line of business, and take into consideration that aggregate stop-loss contracts may be associated with higher parameter uncertainty and moral hazards. This may imply that a higher λ value and a large parameter adjustment *b* need to be used.

Pricing Excess-of-Loss Insurance Contracts

For *excess* business, we need more data than for ground-up (or primary) business. In addition to the required data for ground-up business, we need the following:

- A severity curve based on industry data or theoretical loss distributions.
- Loss payment pattern for the excess cover with an average duration D_{XOL}, which is generally longer than the ground-up payment duration.

From the perspective of a top-down approach, this involves an allocation of overall risk load to various layers. We apply the Wang Transform to the severity curves to derive risk load relativity by layer. If we fix our base layer as (0, 1M), we can calculate a relativity factor for any layer (a,b] as follows:

$$layer\_relativity = \frac{relativty\_loading\_for\_layer(a,b]}{relative\_loading\_for\_base\_layer}$$

We price the excess of loss layer as follows:

(1). Calculate the discount factor $\mathrm{PV}_{\mathrm{XOL}}(1)$: Use market risk-free interest rate and the *excess-layer* loss payment pattern.
(2). Apply risk loading to get a pure premium:

$$PV_{XOL}(1)\cdot\{1+\lambda_1\cdot\sigma_1\cdot D_{XOL}\cdot(layer\_relativity)\}.$$

The factor λ_1 should be calibrated from "total portfolio re-balancing" based on a target return-on-equity (TROE). In other words, for the aggregate insurance portfolio, the ratio of "the total risk load plus investment income" to the total economic capital should produce a target return-on-equity.
The total allocated capital over the lifetime of this $1 liability is

$$\lambda_1\sigma_1 D_{XOL}(layer\_relativity)(1+r)/(TROE-r)\,,$$

For year j, the allocated capital is

$$\lambda_1\sigma_1(layer\_relativity)P_j(1+r)/(TROE-r)\,,$$

where P_j is the expected percentage of excess-layer losses to be paid in year j.
(3). Load for expenses: suppose the total expense factor is θ, we can load the pure premium by a factor of $1/(1-\theta)$.
(4). Knowing the amount of allocated capital, we can calculate the actual ROE for any given quoted premium rate.

Pricing Variable-Rating Excess-of-Loss Insurance Contracts

For variable rating terms such as sliding scales and retrospective rating, we can make some adjustments for the magnitude of risk-reduction. This can be accomplished by applying the Wang Transform to the Lee diagram (see Lee, 1988). We will leave this adjustment method for future discussions.

Pricing of Property Catastrophe Covers

Although the geometric Brownian motion assumption might apply to the evolution of liability losses, it probably does not apply to property catastrophe losses. Here the loss emergence may follow some jump process contingent upon the occurrence of some events.

For property catastrophe covers, modern CAT modeling techniques often use a "bottom-up" perspective. Given geographic spread and amount of insurance data, commercial CAT models can provide us with a final aggregate loss distribution for any given layer of coverage. This final loss distribution already takes into account the potential frequency and severity of CAT events, as well as the correlation (concentration) of the book of business. Ideally, pricing of CAT covers should be based on such "bottom-up" information.

If a final aggregate CAT loss distribution is available, we can apply the Wang Transform directly to it. Theoretically, the parameter λ would be much higher than for the non-CAT counterpart, to reflect a higher correlation risk. Due to the nature of the CAT loss modeling, parameter uncertainty is present and should be reflected in a higher value of *b*.

Here are some additional remarks:

- In most cases we suggest the use of top-down approach, which utilizes industry data by line of business (loss volatility, severity curve, loss payment pattern). The top-down approach is based on the principle of CAPM. In other words, with the top-down approach, only the systematic risk (including the cost of capital) is priced into the contracts. Bault (1995) argued why industry data, rather than individual company data, should be used for pricing purposes.
- In some situations, a bottom-up approach is warranted, especially for pricing property catastrophe covers, where a final loss distribution to the CAT layer can be obtained from available CAT-Models.

- The outlined approach is based on the pricing framework using the Wang Transform. For pricing ground-up business, the Wang Transform extends the classic CAPM in that the parameter λ can now be calibrated from overall industry capital requirements. For pricing excess-of-loss layers, the Wang Transform implies risk-load relativity by layer, in parallel to the Black-Scholes formula for pricing options. For both primary and excess layers, the Wang Transform prescribes a method to account for the duration of liabilities.

- The above calculations did not account for federal income tax. But tax calculations can be incorporated.

Loss Reserve Discounting

Consider the loss reserve liability for a given line of business. The pricing approach can be equally applied to valuation of reserve liabilities. It should be kept in mind that the reserving risk, in terms of $\sigma(t)$, may differ from the pricing risk. Here we provide an alternative (and more direct) approach to the discount of loss reserves.

Again we assume that the loss reserve evolution follows a geometric Brownian motion. For \$1 loss reserve liability, the incurred losses at time T has a distribution

$$X(T) \;\sim\; \text{logormal}\left(\mu T - 0.5\sigma_1\sqrt{T}, \sigma_1^2 T\right).$$

Assume that the risk-free rate is a constant r. The present value of the incurred losses has a distribution:

$$X(T)\exp(-\text{rT}) \;\sim\; \text{logormal}\left((\mu - r)T - 0.5\sigma_1\sqrt{T}, \sigma_1^2 T\right).$$

Let λ_1 be the market price of risk for this line of business with one-year time horizon. For time horizon T, the market price of risk should be $\lambda_1\sqrt{T}$. Applying the Wang Transform to the distribution for the discounted reserves, we get another lognormal distribution:

$$X^*(T)\exp(-\text{rT}) \;\sim\; \text{logormal}\left((\mu - r + \lambda_1\sigma_1)T - 0.5\sigma_1\sqrt{T}, \sigma_1^2 T\right). \qquad (9)$$

From relation (9) we infer that applying the Wang Transform is equivalent to using the following discount rate:

$$i = r - \lambda_1\sigma_1. \qquad (10)$$

In relation to equation (10) we make the following observations:

- The discount rate in equation (10) is the mirror formula of CAPM for assets. It is also in line with a reserve-discount formula proposed by Butsic (1988) and D'Arcy (1988).
- The per annum volatility σ_1 for product liability should be higher than that for worker's compensation. As a result, a lower discount rate should be used for product liabilities.
- For Worker's Compensation lifetime-pension cases, the per annum volatility σ_1 should be negligible and the discount rate should be close to the risk-free rate.
- With this outlined approach, the key parameter λ_1 can be (and should be) calibrated from aggregate industry capital allocations for each sector of underwritten business. This is in contrast to the traditional CAPM method where underwriting beta is derived from running linear regressions of equity prices of insurance firms. The "cost-of-capital" calibration of λ_1 should be more robust than the traditional estimation of underwriting beta.

Final Comments of Part Three

Most of the applications shown are equally applicable to banks and other financial institutions.

Our approach is mainly a top-down approach, which is consistent with CAPM. The top-down approach uses industry aggregate data, rather than relying solely on individual risk distributions. The top-down approach also eliminates any possible inconsistencies related to the treatment of frequency/severity (see Venter, 1998).

In the CAS White Paper on Fair Valuing Property/Casualty Insurance Liabilities, several methods of estimating risk adjustments are surveyed and compared. The White Paper discussed the advantages and disadvantages of the "Distribution Transform Method", including the PH-transform method. The Wang Transform can overcome most of the disadvantages listed in the White Paper:

- As shown earlier, the Wang Transform can be used for producing prices or risk loads on primary business. In fact, under some common assumptions, the Wang Transform reproduces the CAPM method and the Risk-Adjusted Discounting Method, which have both been used in pricing primary business.
- Unlike many other possible transforms including the PH-transform, the simple and more generalized families of the Wang Transform are probably the only ones that builds directly upon CAPM and Black-Scholes Theory.
- The key parameter in the Wang Transform is the market price of risk. This has been a familiar concept to financial economists. Better yet, the market price of risk in the Wang Transform is not subject to the same drawbacks of "underwriting beta." This is because that the market price of risk can be easily calibrated from industry capital requirements. This calibration is more robust than historical estimates of the "underwriting beta".

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