EQUILIBRIUM PRICING TRANSFORMS: NEW RESULTS OF BUHLMANN'S 1980 ECONOMIC MODEL*

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ABSTRACT

In this paper we revisit an economic model of Buhlmann (ASTIN Bulletin, 1980) and derive equilibrium pricing transforms. We obtain the Esscher Transform and the Wang Transform under different sets of assumptions on the aggregate economic environment.

KEYWORDS AND PHRASES

Equilibrium Pricing, Esscher Transform, Wang Transform, Distorted Probability, Exponential Tilting, Optimal Exchange.

ACKNOWLEDGMENTS

The author thanks Hans Buhlmann, Daniel Heyer, Leigh Halliwell, Stephen Mildenhall, and Virginia Young for helpful comments. All errors and omissions remain to be the author's.

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* This paper is dedicated to Dr. Hans Buhlmann for his tremendous contributions to the actuarial profession and the international actuarial community.

1. INTRODUCTION

In the actuarial research literature, there have developed many probability transforms for pricing financial and insurance risks. Since the pricing of risk is always done in an economic/market environment, it is theoretically desirable to derive pricing transforms from a sound economic model that reflects the collective risk preferences of the market participants. Dr. Hans Buhlmann, in his milestone paper published in 1980 ASTIN Bulletin, has developed such an economic model.

Buhlmann argued that in real-life situations premiums are not only depending on the risk to be covered but also on the surrounding market conditions. He defined an economic premium principle as

$$H: (X, Z) \rightarrow \text{Price}[X],$$

where Z represents the market condition (e.g., aggregate risk, collective wealth, correlation, etc).

With the goal of developing a sound economic premium principle, Buhlmann considered a risk-exchange model where all individual agents are acting to maximize his/her own expected utility. Buhlmann's risk-exchange model has roots in mathematical economics.

Under a set of assumptions on the aggregate economic environment, Buhlmann derived equilibrium premiums as those obtained from the Esscher Transform, which is a simple exponential tilting of the probability density: $f^*(x) = c \cdot f(x) \cdot \exp(l x)$, where c is a re-scaling constant. The Esscher Transform has shown tremendous successes in pricing options, see Gerber and Shiu (1994) and Buhlmann et al. (1998).

In another major line of research, Venter (1991) made an observation that insurance prices by (excess-of-loss) layer imply a transformed distribution. This inspired Wang (1995, 1996) to propose premium calculation by applying a distortion to the cumulative distribution function:

$$F^*(x)=g[F(x)],$$

where g:[0,1]→[0,1] is an increasing function with $g(0)=0$ and $g(1)=1$. Among the distortion family, the proportional hazards (PH) transform is widely known to actuaries, partially due to its simplicity. A newly emerged distortion, the Wang Transform, extends CAPM for underlying assets and Black-Scholes formula for options, which has brought the line of research on distortion to a new territory bordering with financial economics. In this paper we shall discover how the distortion approach is related to Buhlmann's equilibrium pricing model.

In sections 2, we revisit the economic model of Buhlmann and derive equilibrium pricing transforms. We obtain the Esscher Transform and the Wang Transform from the equilibrium model, but under distinct sets of assumptions regarding the aggregate economic environment. By focusing on assumptions underlying these pricing transforms, we gain insights about their differences and connections.

In section 3 we discuss "general exponential tilting" to further explore Buhlmann's results. We show how the exponential tilting is related to the distortion pricing approach.

In section 4 built upon Buhlmann's results, we discuss how systematic risks can be reflected by a distortion function.

In section 5 we give interpretations for the general economic model of Buhlmann (1984).

2. BUHLMANN'S EQUILIBRIUM-PRICING MODEL

Consider risk exchanges among a collective of agents j=1, 2, …, n, (typically reinsurers, insurers, buyers of direct insurance, etc).

Each agent is characterized by his/her

(i) utility function $u_j(x)$, with $u_j'(x)>0$, and $u_j''(x)\le 0$;

(ii) initial wealth W_j.

Each agent j is facing a risk of potential loss $X_j(\omega)$ and is buying a risk-exchange $Y_j(\omega)$, where ω represents a state in a probability space (Ω, P). If agent j is an insurance company, we can think of $Y_j(\omega)$ as the sum of all (re)insurance policies bought and sold by j as if it were "one" contract.

Whereas the original risk X_j belongs to agent j, the risk exchange Y_j can be freely bought/sold by agent j in the market. Buhlmann introduced the concept of pricing density
$\phi(\omega)$ such that

$$\text{Price}[Y_j] = \int_\Omega Y_j(w) f(w) d\text{P}(w)\,, \tag{2.1}$$

Buhlmann pointed out that the pricing density $\phi(\omega)$ could be understood as an alteration of the actuarially objective probabilities.

Definition 2.1: The pair $\{Y_{e,j}$ and $f_e\}$ are called in *equilibrium* if

(C-1). For all j, $E\left[u_j\left(W_j - X_j + Y_{e,j} - \text{Price}[Y_{e,j}]\right)\right]$ is maximum among all possible choices of the exchange variables Y_j.

(C-2). $\sum_{j=1}^{n} Y_{e,j}(w) = 0$ for all ω in Ω.

In the equilibrium, $Y_{e,j}$ is called the *equilibrium exchange*, and f_e the *equilibrium price density*.

Theorem 2.1 [Buhlmann, 1980] Assume that each agent j has an exponential utility function $u_j(x) = 1 - \exp(-l_j x)$, the equilibrium price density satisfies:

$$f_e(w) = \frac{\exp(\,lZ(w))}{E[\exp(\,lZ)]}\,, \tag{2.2}$$

where

$$Z(w) = \sum_{j=1}^{n} X_j(w) \tag{2.3}$$

is the aggregate risk, and λ satisfies

$$\frac{1}{\lambda} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \dots + \frac{1}{\lambda_n}. \tag{2.4}$$

From Theorem 2.1, the equilibrium price for any risk X is

$$H_{Buhlmann}[X, \lambda] = \frac{E[X \cdot \exp(\lambda Z)]}{E[\exp(\lambda Z)]}, \tag{2.5}$$

with Z in equation (2.3) and λ in equation (2.4).

Buhlmann (1980) further assumed that *X* and *Z–X* are independent, and derived that

$$H_{Buhlmann}[X, \lambda] = \frac{E[X \cdot \exp(\lambda X)] \cdot E[\exp(\lambda (Z - X))]}{E[\exp(\lambda X)] \cdot E[\exp(\lambda (Z - X))]} = \frac{E[X \cdot \exp(\lambda X)]}{E[\exp(\lambda X)]}. \tag{2.6}$$

Theorem 2.2 Under the set of assumptions:

(AS-1a): The insurance market contains a small number of agents, and

(AS-1b): Individual risk *X* is independent from *Z–X*, where *Z* is the aggregate risk,

the equilibrium price in equation (2.6) is the same as that obtained from the Esscher Transform:

$$f^*(x) = \frac{f(x)\exp(\lambda x)}{E[\exp(\lambda X)]}.$$

Now we examine more carefully the assumptions underlying the derivation of the Esscher Premium in equation (2.6).

For an insurance market with a large number of agents (policy-holders, insurers and reinsurers), the size of an individual risk *X* is negligible relative the industry aggregate loss *Z*. According to equation (2.4), the parameter λ will be close to zero. Using equation (2.6) we get

$$\lim_{\lambda \to 0+} H_{Buhlmann}[X, \lambda] = \lim_{\lambda \to 0+} \frac{E[X \cdot \exp(\lambda X)]}{E[\exp(\lambda X)]} = E[X].$$

For an insurance market in which any individual risk is negligible relative to the size of the aggregate risk, under the assumption that X and Z–X are independent, the equilibrium premium for risk X equals the expected loss without risk loading.

To avoid the complexity of dealing with infinitely large Z and infinitely small λ, it is useful to re-scale Z to $Z_0 = (Z–E[Z])/\sigma[Z]$ and rewrite (2.5) into the following:

$$H_{Buhlmann}[X, l_0] = \frac{E[X \cdot \exp(l_0 Z_0)]}{E[\exp(l_0 Z_0)]}. \tag{2.7}$$

Note that Z_0 has mean=0 and variance=1. For the re-scaled aggregate risk Z_0, the parameter λ_0 represents the market price per unit of risk.

To carry on the analysis of Buhlmann (1980), we make the following set of revised assumptions:

(AS-2a). In aggregate, the total loss Z has a normal distribution, thus the re-scaled variable $Z_0 = (Z–E[Z])/\sigma[Z]$ has a standard normal distribution Φ.

(AS-2b). For risk X with cdf $F(x)$, there exists a standard normal variable V such that $X=F^{-1}(\Phi(V))$, and $\{V, Z_0\}$ have a bivariate normal distribution with correlation coefficient ρ.

Remarks:

- Assumption (AS-2a) is reasonable for an insurance market in which (i) there are a large number of agents and uncorrelated risks, and (ii) each individual risk is negligible in size relative to the aggregate industry risk.
- Assumption (AS-2b) is a direct extension of the multivariate normal assumption used in the derivation of CAPM. For risks with general marginal distributions, here we are assuming a normal-copula correlation structure between X and Z (see Wang, 1998; Frees and Valdez, 1998; Embrechts et al. 1999).

Based on assumption (AS-2b), there exists a normal variable Y independent of X such that $Z_0 = r \cdot V + Y$. Taking $Z_0 = r \cdot V + Y$ into equation (2.7), and using the independence between X and Y, we have

$$H_{Buhlmann}[X, l] = \frac{E[X \cdot \exp(l_0 r \cdot V)] \cdot E[\exp(l_0 Y)]}{E[\exp(l_0 r \cdot V)] \cdot E[\exp(l_0 Y)]},$$

which further leads to

$$H_{Buhlmann}[X, l] = \frac{E[X \cdot \exp(lV)]}{E[\exp(lV)]}, \quad \text{where } \lambda = \rho\lambda_0. \tag{2.8}$$

Theorem 2.3 Under the set of assumptions in (AS-2a) and (AS-2b), the equilibrium premium in equation (2.8) is identical to that obtained by the Wang Transform

$$F^*(x) = \Phi\left[\Phi^{-1}(F(x)) - l\right], \tag{2.9}$$

with $\lambda = \rho\lambda_0$.

Proof: See Section 3, Example 3.1.

Recall that λ_0 represents the aggregate market price per unit of risk and ρ is the correlation coefficient between the normalized variables V and Z_0. The relation $\lambda = \rho\lambda_0$ is a generalization of the classic CAPM to risks with general probability distributions (see Wang, 2000, 2001).

Remark 1. The Wang Transform is a newly emerged distortion function among the distortion family that includes the PH-transform. Under a set of axioms, Wang, Young and Panjer (1997) showed that all coherent risk measures can be represented by a distortion. Among the family of distortions, only the Wang Transform can recover CAPM for underlying assets and Black-Scholes formula for options.

Remark 2. As noted in Buhlmann (1984), the main result in Theorem 2.1 is still valid under general utility function assumptions for the participants. Therefore, under the assumptions (AS-2a) & (AS-2b), Theorem 2.3 effectively gives an independent derivation of CAPM.

Remark 3. The correlation between risk *X* and the aggregate portfolio risk *Z* is the main driver for risk load. The relation $\lambda=\rho\lambda_0$ is rather intuitive since highly correlated risks demand higher risk loading, such as natural or man-made catastrophe risks. In practice, the meaning of correlation should be interpreted more broadly than the statistical association in the claim generating process. From an insurer's perspective, the correlation between profits of insurance contracts is equally important as the correlation between losses. Parameter uncertainty, pricing cycle, and regulatory capital requirements all contribute to the correlation between profits.

3. EXPONENTIAL TILTING & DISTORTION TRANSFORM

To explore further Buhlmann's main results in equation (2.2) & (2.5), we define a general exponential tilting and discuss its connections with the distortion transform.

Consider variables *X* and *Z* in a probability space (Ω, P) with probability distributions *F* and *Q*, respectively.

Definition 3.1 The transformed probability (density) function

$$f^*(x) = f(x)\cdot\frac{E[\exp(lZ)\mid X=x]}{E[\exp(lZ)]}, \tag{3.1}$$

is called an exponential tilting of *X*, induced by *Z*.

As special cases of Definition 3.1, when $Z = X$ we recover the Esscher Transform; When $Z = h(X)$ is an increasing function of *X*, we recover the generalized Esscher transform by Heilmann (1989), including the special case of $h(X)=1-\exp(-\lambda X)$ in Kamps (1998).

For any probability distribution $F(x)$, we define its inverse function as

$$F^{-1}(u) = \sup\{x : F(x) < u\},\ \text{ for } 0\le u \le 1.$$

Assume that there exists a uniform random variable U such that $X=F^{-1}(U)$ and $Z=Q^{-1}(U)$. We say that X and Z are co-monotone. Here "co-monotone" means perfect correlation, which extends beyond the concept of perfect linear correlation. There is no diversification benefit between "co-monotone" risks, see Wang and Dhane (1998).

When X and Z are co-monotone, it may be impossible to express Z as a direct function of X. For example, consider the case that X has a Bernoulli distribution and Q is an exponential distribution.

Theorem 3.1 When X and Z are co-monotone, the exponential tilting in equation (3.1) implies a transform: $F(x) \rightarrow F^*(x)$ by

$$F^*(x) = \frac{1}{M_Q(l)} \int_0^{F(x)} \exp\left(l \cdot Q^{-1}(u)\right) du\,, \tag{3.2}$$

where

$$M_Q(l) = \int_0^1 \exp\left(l \cdot Q^{-1}(u)\right) du$$

exists for some λ>0.

- When F is continuous at x, the transformed probability density at x is

$$f^*(x) = \frac{f(x) \cdot \exp\left(l \cdot Q^{-1}(F(x))\right)}{M_Q(l)}. \tag{3.3}$$

- When F is discrete on $\{x_1, x_2, \ldots, x_m\}$, the transformed probability at x_j is

$$f^*(x_j) = \frac{1}{M_Q(l)} \int_{F(x_{j-1})}^{F(x_j)} \exp\left(l \cdot Q^{-1}(u)\right) du\,. \tag{3.4}$$

We call equation (3.2) a *co-monotone exponential tilting*, induced by the *kernel Q*.

Consider an important case when the kernel Q is unrelated to the distribution F.

Theorem 3.2 When the kernel Q is independent of the distribution F, the co-monotone exponential tilting (3.2) is equivalent to a distortion $F^*(x)=g(F(x))$ with

$$g(u) = \frac{\int_0^u \exp(l \cdot Q^{-1}(v))dv}{M_Q(l)}. \tag{3.5}$$

Proof: The key here is that Q is independent of F. It then follows directly from the co-monotone exponential tilting equation (3.2).

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Example 3.1 When the kernel Q=Φ is the standard normal distribution, the co-monotone exponential tilting (3.2) recovers the Wang Transform:

$$F^*(y) = \Phi\left(\Phi^{-1}(F(y)) - l\right).$$

Example 3.2 When the kernel $Q(t)$=1–exp(–t), for t≥0, is an exponential distribution, the co-monotone exponential tilting (3.2) recovers the proportional hazards (PH) transform as introduced in Wang (1995):

$$F^*(x) = 1 - (1 - F(x))^{1-l}, \text{ with } 0 \leq l < 1,$$

which corresponds to the distortion $g(u) = 1-(1-u)^{1-l}$.

This result gives an interpretation of the parameter l in the PH-transform.

Example 3.3 When the kernel $Q(t) = t$, for $0 \leq t \leq 1$ is a uniform distribution, the co-monotone exponential tilting (3.2) recovers the exponential distortion:

$$g(u) = \frac{\exp(lu) - 1}{\exp(l) - 1}, \text{ for } 0 < u \leq 1.$$

Example 3.4 When the kernel $Q(z)$ =Gamma(z; α, β) has a gamma distribution with mean=α/β, the co-monotone exponential tilting (3.2) corresponds to the following distortion:

$$g(u) = \text{Gamma }(Q^{-1}(u); a, b - l).$$

For the distortion in (3.5) we have

$$g'(u) = \exp\left(l \cdot Q^{-1}(u)\right) > 0\ . \tag{3.6}$$

When $Q(x)$ is differentiable, we have

$$g''(u) = l \exp\left(l \cdot Q^{-1}(u)\right) / Q'(Q^{-1}(u)) \geq 0, \qquad \text{for } \lambda > 0. \tag{3.7}$$

For the Beta family of distortion (Wirch and Hardy, 1999)

$$F(x) = 1 - \int_0^{1-F(x)} \frac{t^{a-1}(1-t)^{b-1}}{\Gamma(a,b)} dt$$

to be an exponential tilting with $\lambda>0$, it is necessary that $a\leq 1$ and $b\geq 1$.

Finally we offer a comment on the numerical implementations of co-monotone exponential tilting. Computer calculations almost always use discrete data points. Consider a discrete representation $\{x_1 < x_2 <\ldots <x_m\}$ of variable X, where $F(x_m)=1$. One would need to perform numerical integration to carry out the exponential tilting as shown in equation (3.4). The reader needs to be aware of that the simple approximation by

$$f^*(x_j) \approx f(x_j) \exp\left(l \cdot Q^{-1}(F(x_j))\right) / M_Q(l) \tag{3.8}$$

is often very poor, especially at the tails of the distribution. For the Wang Transform and PH-transform, we have $Q^{-1}(F(x_m))=+\infty$, thus the approximation $f^*(x_m)$ in (3.8) is undefined. Such numerical difficulties can be avoided by applying the distortion in (3.5) directly on the (discrete) cumulative distribution function.

4. SYSTEMATIC RISKS & THE DISTORTION APPROACH

Consider Buhlmann's economic model in section 2, we assume that risk X can be decomposed into two parts

$$X = X_{sys} + X_{non},$$

where

- X_{sys} (being co-monotone with Z) represents the systematic portion of X, and
- X_{non} (being uncorrelated with Z) represents the idiosyncrasy or non-systematic portion.
- By definition, X_{sys} and X_{non} are uncorrelated.

From equations (2.5) & (2.6) we have

$$H_{Buhlmann}[X, l] = E[X_{non}] + \frac{E[X_{sys} \cdot \exp(lZ)]}{E[\exp(lZ)]}. \tag{4.1}$$

In other words, Buhlmann's equilibrium pricing model indicates that only the systematic risk requires risk loading.

For convenience, we assume that the distribution F for a risk X only reflects the systematic risk of X. For practically minded reader, this is quite in agreement with reality. For instance, life insurers are generally not too concerned about the volatility of an individual life contract, but rather more concerned about the systematic errors in their estimate of mortality rates, and systematic shocks. As a result, in the pricing exercise by insurers, only systematic risks enter into the distribution F, manifested in the modeling of potential variations for a large block of contracts, or for a whole line of business, etc.

In light of equation (4.1) we now make the following simplifying assumptions:

(**AS-3a**). All potential variations that are reflected in the distribution F_j are systematic risk only. As a result, risk X_j is co-monotone with the aggregate risk Z.

(**AS-3b**). There are many market participants so that the re-scaled aggregate variable Z_0 =(Z–E[Z])/σ[Z] has a distribution Q which is unrelated to the individual risk distribution F_j.

Theorem 4.1 Under the assumptions in (AS-3a) and (AS-3b), the equilibrium price density in equation (2.2) is identical to the distortion $F^*(x)=g(F(x))$ with

$$g(u)=\frac{\int_0^u \exp(l\cdot Q^{-1}(v))dv}{M_Q(l)}.$$

Specially,

- when Q is an Exponential(1) distribution, we recover the PH-transform;
- when Q is standard normal distribution, we recover the Wang Transform.

5. BUHLMANN'S GENERAL ECONOMIC MODEL

In a follow-up paper, Buhlmann (1984) extended his economic premium principle using general utility functions for each participant. He discovered that all equilibrium prices are locally like the one where agents have exponential utilities; the only difference being that risk aversion is no longer constant but depends on the agents' wealth. His general economic model provides further insights on the relation between risk premium and aggregate market conditions.

Under general utility assumptions, Buhlmann used the notion of absolute risk aversion of Pratt (1964):

$$A_j(x) = -u_j''(x)/\, u_j'(x), \tag{5.1}$$

which depends on the amount of net wealth x. Note that for an exponential utility function $u_j(x) = 1 - \exp(-\lambda_j x)$ we have $A_j(x) = \lambda_j$ (constant).

Buhlmann showed that equilibrium exists under only modest theoretical assumptions. He pointed out that this equilibrium also coincides with the Pareto optimal exchange in Borch (1962).

As an important observation, Buhlmann pointed out that in equilibrium $Y_{e,j}$ and f_e should depend on ω only through $Z(\omega) = \sum_{j=1}^{n} X_j(\omega)$. As a result, Buhlmann introduced a generic element $\eta = Z(\omega)$.

Theorem 5.1 [Buhlmann, 1984] Under general utility assumptions, the equilibrium pricing density satisfies

$$\frac{f_e'(\eta)}{f_e(\eta)} = A(W,\eta)\,, \tag{5.2}$$

where $A(W,\eta)$ is the total risk aversions satisfying

$$\frac{1}{A(W,\eta)} = \sum_{j=1}^{n} \frac{1}{A_j\left[W_j - X_j(\eta) + Y_{e,j}(\eta) - \mathrm{Price}[Y_{e,j}(\eta)]\right]}. \tag{5.3}$$

From equation (5.2) we can see that the local behavior for the equilibrium pricing density is the same as that for the exponential utilities.

Theorem 5.1 provides additional insights to the parameter $\lambda=A(W,\eta)$. As the total risk-aversion of the market, the parameter λ depends on the aggregate wealth (or capital) of the market participants. The presence of excessive capital will drive down the parameter $\lambda=A(W,\eta)$ and the resulting insurance prices. A shortage of capital can boost the level of $\lambda=A(W,\eta)$ and the resulting insurance prices.

The insurance industry has experienced surprises by unexpected catastrophe events; For instance, the huge insurance losses due to 1992 Hurricane Andrew, and the September 11, 2001 Terror Attack on America.

Buhlmann's economic model can explain some of the after effects of unexpected catastrophe events:

a) As a Bayesian update, the estimated probability of loss will increase, especially for large loss amounts.
b) A catastrophe may simultaneously impact many lines of business. This will elevate the perceived correlation between lines of business, and have an effect of increasing the systematic risk for X_j.
c) The market price of risk, $\lambda=A(W, \eta)$ in equation (5.3), will increase because of the depletion of the aggregate wealth after paying for the occurred catastrophe loss.
d) The compounding effect of these factors is a dramatic increase in risk load and premium rates.
e) Because of the increase in the prospective Sharpe ratio $\lambda=A(W,\eta)$ in (5.3), "smart" capital may be injected from the outside to take advantage the increased Sharpe ratio prospect, as evidenced in new entrants to the insurance market after hurricane Andrew & Terror Attack.

6. SUMMARY

For financial and insurance risks, their equilibrium prices will generally depend on assumptions about the utility functions of the market participants. Buhlmann's (1980) economic model is very profound in that equilibrium pricing transforms can be derived under general utility functions of the market participants. Other good reference papers on equilibrium risk-exchanges include Aase (1993), Taylor (1992), Gerber and Pafumi (1998). In a practical context, Meyers (1996) also takes an equilibrium pricing approach to calculating risk load.

There are fundamental differences between option-pricing and insurance pricing (see Mildenhall, 1999). It is remarkable that the Esscher Transform (for option pricing) and the Wang Transform (for insurance pricing) can both be derived from the same optimal risk-exchange model of Buhlmann (1980).

Buhlmann's model indicates that only non-diversifiable risks require risk loading. In practice, there are good reasons to interpret non-diversifiable risks more broadly than loss correlation.

Buhlmann's general economic model (1984) indicates that industry over-capitalization will likely lead to lower risk loading; Likewise, industry under-capitalization will likely lead to higher risk loa ding.

In summary, Buhlamnn's economic model has much to offer to the actuarial profession.

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